UNITARY EQUIVALENCE OF LINEAR TRANSPORT MODELS

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Abstract

One simple unitary transformation is provided between the linear transport model of R. Beals and a model presented elsewhere. Special attention is paid to similar relationships in electron transport theory.

During the past few years the investigation of the abstract differential equation

\[ (T\psi)'(x) = -A\psi(x), \quad 0 < x < \tau(\omega), \]

(1)

where T and A are self-adjoint operators on a Hilbert space \( \mathcal{H} \) and "partial range" boundary conditions are imposed, has grown to be a popular theme. Being the offspring of Hangelbroeks thesis\(^1\) the subject has been subjected to study by e.g. Beals\(^2,3,4\), Hangelbroek\(^5\), Kaper and Lekkerkerker\(^6,7,8\), van der Mee\(^9,10\), and recently Greenberg and Zweifel\(^10\), and has been applied in several branches of physics\(^1,2,6,9,10\). The main purpose of this short note is to present one simple transformation, which yields a short derivation of the main results of Ref.7 from the results of...
Beals\(^3\) and makes explicit the connection between ideas of Refs. 3, 4 and 9 and ideas presented in Ref. 8. This transformation is an extension of \(T\) to a unitary operator.

Let us suppose that \(T\) is bounded with zero null space and \(A\) is positive with closed range. Following Beals\(^3\) we define \(H_T\) to be the completion of \(H\) with respect to the inner product

\[
(x,y)_{|T|} = (|T|x,y) \quad (x,y \in H).
\]

(2)

According to Kaper and Lekkerkerker\(^7,8\) \(H_{T-1}\) is the completion of \(\text{Im} T = \{Tx/ x \in H\}\) with respect to the inner product

\[
(x,y)_{|T|^{-1}} = (|T|^{-1}x,y) \quad (x,y \in \text{Im} T).
\]

(3)

Clearly \(T\) extends to a unitary operator from \(H_T\) onto \(H_{T-1}\)\(^11\), which establishes a natural relationship between half-space and finite slab results of Ref. 3 (formulated in \(H_T\)) and their analogues of Ref. 7 (formulated in \(H_{T-1}\)): any operator \(K\) of Ref. 3 (such as the Larsen-Habetler\(^12\) albedo operator) is connected to its analogue \(K^*\) of Ref. 7 by the formula

\[
TK = K^*T : H_T \to H_{T-1}.
\]

(4)

The transformation \(T\) is readily suggested on comparing Eqs. (3.8), (3.9) and (3.11) of Ref. 3 to Eqs. (4.4) and (4.5) and the bottom formula at page 358 of Ref. 7. In this way the Beals\(^3\) solvability results for the half-space and finite slab problems (formulated through \(H_T\)) can be transformed into those of Kaper and Lekkerkerker\(^7\), as exemplified by Lemma 3.2 in combination with Lemma 3.1 of Ref. 3 versus the invertibility of \(V\) and \(V_T\) in Section 4 of Ref. 7.

In Ref. 8 the electron transport half-space problem is stated and some ideas for its solution are considered worth presenting.
In this problem A has a one-dimensional null space and has a compact resolvent. Denoting the (two-dimensional) zero generalized eigenvector spaces of $T^{-1}A$ and $AT^{-1}$ by $H_0$ and $H_0^+$, respectively, one finds

$$T[H_0] = H_0^+ \subset \text{Im } T.$$  \hspace{1cm} (5)

In this way $T$ acts as a unitary operator from $H_0$ onto $H_0^+$, provided one endows $H_0$ with the indefinite inner product$^{13)$

$$(x,y)_T = (T^* x, y)$$ \hspace{1cm} (6)

and $H_0^+$ with the indefinite inner product$^{13)$

$$(x,y)_{T^{-1}} = (T^{-1} x, y).$$ \hspace{1cm} (7)

Maximal positive/negative subspaces of $H_0$ (with respect to (6)) are mapped by $T$ onto maximal positive/negative subspaces of $H_0^+$ (with respect to (7)). Half-space problems with non-injective $A$ were studied rigorously in Refs. 6, 3 and 9. The idea to exploit the indefinite inner product (6) to solve half-space problems with non-injective $A$ was first published by van der Mee.$^9$ For electron transport a parallel idea, through the "$T$-transform" (7) of (6), turned up in Ref. 8 together with the suggestion to study "connecting" transformations on $H_{T^{-1}}$. Though in a not completely correct way$^{10}$, "$T^{-1}$-transforms" of such connecting transformations were investigated before by Beals$^9$. Again a unitary extension of $T$ could be applied to make the connection between different papers.

The aforementioned relationships are based on a general principle. For neutron transport the solution $\psi(x)$ of Eq. (1) represents a neutron angular density, whereas $T \psi(x)$ represents a current density$^{14}$. In radiative transfer a similar pair of physical concepts is involved, namely the intensity and radiative flux. One could say that $T$ transforms angular densities (resp. intensities) into current densities (resp. radiative fluxes). In both applications $T$ is the multiplication operator by the cosine of the direction of propagation.
References

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