SIAM J. Matrix Anal. Appl.
Vol. 22, No. 4, pp. 1263–1273

REAL HAMILTONIAN POLAR DECOMPOSITIONS*

CORNELIS V. M. VAN DER MEE†, ANDRÉ C. M. RAN‡, AND LEIBA RODMAN§

Abstract. For a given real invertible skew-symmetric matrix H, we characterize the real $2n\times 2n$ matrices X that allow an H-Hamiltonian polar decomposition of the type $X = UA$, where U is a real H-symplectic matrix ($U^T HU = H$) and A is a real H-Hamiltonian matrix ($HA = -A^T H$).

Key words. polar decomposition, Hamiltonian matrices, skew-Hamiltonian matrices, symplectic matrices

AMS subject classifications. 15A23, 47B50

PII. S0895479899362788

1. Introduction.

It is well known that every square matrix X allows a polar decomposition $X = UA$, where U is unitary and A is self-adjoint, and the proof of this fact is straightforward. When unitarity and self-adjointness are required to hold with respect to the indefinite scalar product $[x, y] = \langle Hx, y\rangle$ with H an invertible self-adjoint matrix, the theory of the H-polar decompositions $X = UA$, where U is H-unitary (i.e., $[Ux, Uy] = [x, y]$ for all vectors x, y) and A is H-self-adjoint (i.e., $[Ax, y] = [x, Ay]$ for all vectors x, y), is much more complicated and has been developed in [2, 3, 4]. Introducing the H-adjoint $X^{[*]}$ of X by $X^{[*]} = H^{-1}X^*H$ with X^* the usual adjoint (so that U is H-unitary if and only if U is invertible and $U^{-1} = U^{[*]}$, and A is H-self-adjoint if and only if $A^{[*]} = A$), an H-polar decomposition of a matrix X exists if and only if there exists an H-self-adjoint matrix A satisfying

$$X^{[*]}X = A^2, \quad \operatorname{Ker} X = \operatorname{Ker} A, \tag{1.1}$$

where the symbol Ker denotes the null space of a matrix. The H-unitary factor U is then constructed as an H-unitary extension (a so-called Witt extension) of the H-isometry $V : \operatorname{Im} A \to \operatorname{Im} X$ satisfying $Xy = VAy$ for every vector y. An H-polar decomposition of a given matrix X need not always exist, X may have many "nonequivalent" H-polar decompositions, and there exist various interesting subclasses of H-polar decompositions. Moreover, there exists a fairly complete stability theory for H-polar decompositions [6].

The situation is quite different for Hamiltonian polar decompositions, introduced below. Dealing exclusively with real matrices, we fix an invertible $2n \times 2n$ real matrix H such that $H = -H^T$. Without loss of generality we may assume that

$$H = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

*Received by the editors October 28, 1999; accepted for publication (in revised form) by P. Van Dooren October 9, 2000; published electronically April 6, 2001.
http://www.siam.org/journals/simax/22-4/36278.html

†Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy (cornelis@krein.unica.it). The work of this author was partially supported by INDAM and MURST.

‡Divisie Wiskunde en Informatica, Faculteit Exacte Wetenschappen, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands (ran@cs.vu.nl).

§Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187-8795 (lxrodm@math.wm.edu). The work of this author was partially supported by NSF grant DMS 9800704 and by a Faculty Research Assignment grant from the College of William and Mary.

A real $2n \times 2n$ matrix X is called *H-Hamiltonian* if $HX = -X^T H$, and *H-skew-Hamiltonian* if $HX = X^T H$. Denoting $X^{[*]} = H^{-1}X^T H$, we see that X is H-Hamiltonian if and only if $X^{[*]} = -X$, and H-skew-Hamiltonian if and only if $X^{[*]} = X$. Defining a real matrix U to be *H-symplectic* if $U^T HU = H$ or, equivalently, $U^{[*]} = U^{-1}$, and *H-antisymplectic* if $U^T HU = -H$ or, equivalently, $U^{[*]} = -U^{-1}$, we can in principle study four different polar decomposition problems for a given real $2n\times 2n$ matrix X, namely, we can study the problem of representing such X in the form $X = UA$, where U is H-symplectic (or H-antisymplectic) and A is H-Hamiltonian (or H-skew-Hamiltonian). In this article we will limit ourselves to *H-Hamiltonian polar decompositions* only, i.e., to representations of X of the type $X = UA$, where U is H-symplectic and A is H-Hamiltonian.

All matrices in sections 1 *and* 2 *are assumed to be real.*

The following result is immediate. For the sake of completeness we present a short proof.

THEOREM 1.1. *A real $2n \times 2n$ matrix X has an H-Hamiltonian polar decomposition if and only if there exists an H-Hamiltonian matrix A such that $A^2 = -X^{[*]}X$ and* $\operatorname{Ker} A = \operatorname{Ker} X$.

Proof. The necessity is clear: if $X = UA$ is an H-Hamiltonian polar decomposition, then

$$X^{[*]}X = A^{[*]}U^{[*]}UA = A^{[*]}A = -A^2,$$

and $\operatorname{Ker} A = \operatorname{Ker} X$ holds as well. Conversely, if an H-Hamiltonian matrix A exists with the properties as described in the theorem, then there exists an invertible map $U_0 : \operatorname{Im} A \to \operatorname{Im} X$ defined by the equality $U_0Ay = Xy$ for every $y \in \mathbb{R}^{2n}$. Letting $[x, y] = \langle Hx, y\rangle$ be the skew-symmetric scalar product induced by H, we now have

$$\begin{aligned}[U_0Ax, U_0Ay] &= [Xx, Xy] = [X^{[*]}Xx, y] = [-A^2x, y] \\ &= [A^{[*]}Ax, y] = [Ax, Ay], \quad x, y \in \mathbb{R}^{2n}.\end{aligned}$$

In other words, U_0 is an H-isometry. By a version of Witt's theorem (see Theorem 4.2 of [3]), U_0 can be extended to an H-symplectic linear transformation U on the whole of $\mathbb{R}^{2n}$. Thus, we obviously have an H-Hamiltonian polar decomposition $X = UA$. $\square$

The following result recently proved in [1] greatly simplifies the problem of characterizing the real matrices X having an H-Hamiltonian polar decomposition.

THEOREM 1.2. *Every H-skew-Hamiltonian matrix is a square of an H-Hamiltonian matrix. Moreover, for every H-skew-Hamiltonian matrix A there exists an H-symplectic matrix U such that*

$$U^{-1}AU = \begin{bmatrix} B & 0 \\ 0 & B^T \end{bmatrix} \tag{1.2}$$

for some matrix B. Furthermore, B can be chosen in a real Jordan form.

Every matrix of the form $X^{[*]}X$ is obviously H-skew-Hamiltonian. The converse is also true, as stated in the following result. Proposition 1.3 is to be contrasted with the corresponding results for the symmetric (in the real case) or Hermitian (in the complex case) indefinite scalar products (see [6]): There the three classes of matrices A for which $A = A^{[*]}$, or $A = X^{[*]}X$ for some X, or $A = X^{[*]}X$ for some X such that $\operatorname{Ker} X = \operatorname{Ker} A$, are all different.

Proposition 1.3. *Let A be H-skew-Hamiltonian. Then there exists a matrix X such that $A = X^{[*]}X$ and* $\operatorname{Ker} A = \operatorname{Ker} X$.

Proof. By Theorem 1.2 we may (and do) assume that

$$H = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad A = \begin{bmatrix} B & 0 \\ 0 & B^T \end{bmatrix}$$

for some matrix B. We then let

$$X = \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix},$$

where C and G are such that $B = G^T C$. Then the equality $A = X^{[*]}X$ is easily verified.

To ensure the condition $\operatorname{Ker} A = \operatorname{Ker} X$ we need $\operatorname{Ker} C = \operatorname{Ker} B$ and $\operatorname{Ker} G = \operatorname{Ker} B^T$. To this end, write the singular value decomposition $B = UDV$, where U and V are real orthogonal and D is diagonal with nonnegative entries, and put $C = \sqrt{D}V$, $G = (U\sqrt{D})^T$. ◻

Analogously one proves that for every H-skew-Hamiltonian matrix A there exists X such that $A = -X^{[*]}X$ and $\operatorname{Ker} A = \operatorname{Ker} X$.

Proposition 1.4. *Every invertible $2n \times 2n$ matrix X has an H-Hamiltonian polar decomposition.*

Proof. By Theorem 1.2, there exists an H-Hamiltonian matrix A such that $A^2 = -X^{[*]}X$. Since X is invertible, the condition $\operatorname{Ker} A = \operatorname{Ker} X$ is trivially satisfied. By Theorem 1.1, we are done. As a matter of fact, the H-symplectic factor is given by $U = XA^{-1}$. ◻

There are examples of matrices that do not have any H-Hamiltonian polar decompositions.

Example 1.5. Let

$$H = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$$

be 4×4, where I_2 stands for the identity matrix of order 2. The matrix

$$W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is H-skew-Hamiltonian, so by Proposition 1.3 there exists X such that $W = -X^{[*]}X$ and $\operatorname{Ker} W = \operatorname{Ker} X$, for example,

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, a direct verification shows (see below) that there is no H-Hamiltonian square root V of W such that

$$\operatorname{Ker} V = \operatorname{Ker} W. \tag{1.3}$$

By Theorem 1.1, X has no H-Hamiltonian polar decomposition.

To verify that there is no H-Hamiltonian square root V of W with the property (1.3), assume that V is one. Being H-Hamiltonian, V must be of the form

$$V = \begin{bmatrix} E & F \\ G & -E^T \end{bmatrix},$$

where F and G are symmetric 2×2 matrices. Condition (1.3) implies that V must have a first and a last (fourth) column consisting of zeros. Thus, V must have the form

$$V = \begin{bmatrix} 0 & -p & q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r & p & 0 \end{bmatrix}$$

for some $p, q, r \in \mathbb{R}$. But then $V^2 = 0$, a contradiction with $V^2 = W$.

The main result in [1], i.e., Theorem 1.2, allows us to write a short paper. The main result on the characterization of the real matrices X allowing an H-Hamiltonian polar decomposition is stated and proved in section 2. The final section 3 is devoted to a comparison of the main result to existing results on K-polar decomposition for the invertible self-adjoint matrix $K = iH$.

2. Main result. To formulate and prove our main result, we need canonical forms for H-Hamiltonian and H-skew-Hamiltonian matrices, where H is a fixed real invertible skew-symmetric matrix. We will state these forms only to the extent in which they are needed in our proofs. For the complete canonical forms for H-Hamiltonian and H-skew-Hamiltonian matrices, as well as for pairs of matrices with related symmetries, see, e.g., [5, 7].

LEMMA 2.1.

(a) *Let A be H-skew-Hamiltonian. Then there exists an invertible real matrix S such that $\tilde{A} = S^{-1}AS$ and $\tilde{H} = S^THS$ have the following form:*

$$\tilde{A} = \bigoplus_{j=1}^{k} \left(J_j \oplus (J_j)^T\right),$$
$$\tilde{H} = \bigoplus_{j=1}^{k} \begin{bmatrix} 0 & I_{p_j} \\ -I_{p_j} & 0 \end{bmatrix},$$

where J_j is a real Jordan block of size $p_j \times p_j$ corresponding either to a real eigenvalue or to a pair of nonreal complex eigenvalues.

(b) *Let A be H-Hamiltonian. Then there exists an invertible real matrix S such that $\tilde{A} = S^{-1}AS$ and $\tilde{H} = S^THS$ have the following form:*

$$\tilde{A} = \tilde{A}_0 \oplus \left[\bigoplus_{j=1}^{k} J_{2p_j}(0)\right] \oplus \left[\bigoplus_{j=k+1}^{\ell} \left(J_{2p_j+1}(0) \oplus -J_{2p_j+1}(0)^T\right)\right],$$
$$\tilde{H} = \tilde{H}_0 \oplus \left[\bigoplus_{j=1}^{k} \varepsilon_j F_{2p_j}\right] \oplus \left[\bigoplus_{j=k+1}^{\ell} \begin{bmatrix} 0 & I_{2p_j+1} \\ -I_{2p_j+1} & 0 \end{bmatrix}\right],$$

where $\tilde{A}_0$ is invertible, $J_q(0)$ stands for the $q \times q$ nilpotent (upper triangular) Jordan block, ε_j are signs ± 1, and

$$F_{2p} = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix}$$

is the $2p \times 2p$ skew-symmetric matrix with zeros off the trailing diagonal.

We now state our main result.

THEOREM 2.2. *Let X be a real matrix and H a real skew-symmetric invertible matrix. Then there exists an H-Hamiltonian polar decomposition of X if and only if the part of the canonical form of $(X^{[*]}X, H)$, as presented in Lemma 2.1(a), corresponding to the zero eigenvalue of $X^{[*]}X$, can be represented in the block diagonal form*

$$(\operatorname{diag}(B_r)_{r=0}^m, \operatorname{diag}(G_r)_{r=0}^m),$$

where

(i) *B_0 is the zero matrix of order $2k_0$ and*

$$G_0 = \begin{bmatrix} 0 & I_{k_0} \\ -I_{k_0} & 0 \end{bmatrix},$$

(ii) *$m = m_1 + m_2$, and for each $r = 1, \dots, m_1$ we have*

$$B_r = \begin{bmatrix} J_{k_r}(0) & 0 \\ 0 & J_{k_r}(0)^T \end{bmatrix}, \qquad G_r = \begin{bmatrix} 0 & I_{k_r} \\ -I_{k_r} & 0 \end{bmatrix}, \tag{2.1}$$

while for $r = m_1 + 1, \dots, m_1 + m_2$ we have

$$B_r = \begin{bmatrix} J_{k_r}(0) & 0 & 0 & 0 \\ 0 & J_{k_r-1}(0) & 0 & 0 \\ 0 & 0 & J_{k_r}(0)^T & 0 \\ 0 & 0 & 0 & J_{k_r-1}(0)^T \end{bmatrix},$$
$$G_r = \begin{bmatrix} 0 & I_{2k_r-1} \\ -I_{2k_r-1} & 0 \end{bmatrix}, \tag{2.2}$$

(iii) *and, denoting the corresponding basis in $\operatorname{Ker}(X^{[*]}X)^{2n} \subseteq \mathbb{R}^{2n}$ in which the form (i), (2.1), (2.2) is achieved by $\{e_{r,j}\}_{r=0,j=1}^{m,\ell_r}$, where $\ell_0 = 2k_0$, $\ell_r = 2k_r$ for $r = 1, \dots, m_1$, and $\ell_r = 4k_r - 2$ for $r = m_1 + 1, \dots, m_2$, we have*

$$\operatorname{Ker} X = \operatorname{span}\{e_{r,1} + \varepsilon_r e_{r,2k_r} \mid r = 1, \dots, m_1\}$$
$$+\operatorname{span}\{e_{r,1}, e_{r,4k_r-2} \mid r = m_1 + 1, \dots, m_2\}$$
$$+\operatorname{span}\{e_{0,j}\}_{j=1}^{2k_0} \tag{2.3}$$

for some numbers $\varepsilon_r = \pm 1$.

Note that there may be more than one way to divide the part of the canonical form of $(X^{[*]}X, H)$ corresponding to the zero eigenvalue of $X^{[*]}X$ into blocks of the form (i), of the form (2.1), and of the form (2.2). Also, for some bases in which the form (i), (2.1), (2.2) is achieved the formula (2.3) may be valid, and for some other bases in which the form (i), (2.1), (2.2) is achieved the formula (2.3) may not be valid. Theorem 2.2 says that a necessary and sufficient condition for existence of an H-Hamiltonian polar decomposition of X is that *there is* a suitable division of the part of the canonical form of $(X^{[*]}X, H)$ corresponding to the zero eigenvalue of $X^{[*]}X$ into the blocks of the forms (i), (2.1), and (2.2), and *there is* a suitable basis in which this division is achieved so that (2.3) is valid.

Proof. First of all we show that the proof can be reduced to the case when $X^{[*]}X$ is nilpotent.

By Lemma 2.1 we may let

$$X^{[*]}X = S^{-1}\begin{bmatrix} Z_1 & 0 \\ 0 & Z_0 \end{bmatrix} S, \qquad H = S^T \begin{bmatrix} H_1 & 0 \\ 0 & H_0 \end{bmatrix} S,$$

where Z_1 is invertible and Z_0 is nilpotent. Replacing X by $S^{-1}XS$ and H by S^THS, we see that we may assume without loss of generality that

$$X^{[*]}X = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & H_0 \end{bmatrix},$$

with Z_1 invertible and Z_0 nilpotent and Z_i being H_i-skew Hamiltonian for $i = 0, 1$. So, for the sake of the present argument, we shall assume that this is the case. Then

$$(X^{[*]}X)^n = \begin{bmatrix} Z_1^n & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that

$$\mathbb{R}^{2n} = \mathrm{Im}\,(X^{[*]}X)^n \oplus \mathrm{Ker}\,(X^{[*]}X)^n. \tag{2.4}$$

Note also $\mathrm{Ker}\,X \subseteq \mathrm{Ker}\,(X^{[*]}X)^n$. Define $\tilde{X}$ as follows: $\tilde{X}x = 0$ for $x \in \mathrm{Im}\,(X^{[*]}X)^n$, while $\tilde{X}x = Xx$ for $x \in \mathrm{Ker}\,(X^{[*]}X)^n$. It follows that

$$\mathrm{Ker}\,\tilde{X} = \mathrm{Im}\,(X^{[*]}X)^n \oplus \mathrm{Ker}\,X \tag{2.5}$$

with respect to decomposition (2.4). Indeed, the inclusion $\supseteq$ in (2.5) is obvious in view of the definition of $\tilde{X}$. For the opposite inclusion, let $x + y \in \mathrm{Ker}\,\tilde{X}$, where $x \in \mathrm{Im}\,(X^{[*]}X)^n$, $y \in \mathrm{Ker}\,(X^{[*]}X)^n$. Then clearly $y \in \mathrm{Ker}\,\tilde{X}$, and hence $y \in \mathrm{Ker}\,X$ by the definition of $\tilde{X}$. Note also the equality

$$\tilde{X}^{[*]}\tilde{X} = \begin{bmatrix} 0 & 0 \\ 0 & Z_0 \end{bmatrix}. \tag{2.6}$$

To verify (2.6), first note that because of (2.5), $\tilde{X}^{[*]}\tilde{X}$ has the form

$$\begin{bmatrix} 0 & ? \\ 0 & ? \end{bmatrix},$$

and the H-skew-Hamiltonian property of $\tilde{X}^{[*]}\tilde{X}$ implies that in fact

$$\tilde{X}^{[*]}\tilde{X} = \begin{bmatrix} 0 & 0 \\ 0 & ? \end{bmatrix},$$

the question marks denoting irrelevant parts of matrices. For every $x, y \in \operatorname{Ker}(X^{[*]}X)^n$ we have

$$\langle \tilde{X}^{[*]}\tilde{X}x, y\rangle = \langle H^{-1}\tilde{X}^T H\tilde{X}x, y\rangle,$$

which in view of the definition of $\tilde{X}$ is equal to

$$-\langle HXx, \tilde{X}H^{-1}y\rangle = -\langle HXx, XH^{-1}y\rangle = \langle X^{[*]}Xx, y\rangle.$$

Therefore, the lower right block of $\tilde{X}^{[*]}\tilde{X}$ must be Z_0, as claimed by equality (2.6).

Now assume that X has an H-Hamiltonian polar decomposition $X = UA$. Then A commutes with $X^{[*]}X = -A^2$, and since Z_1 and Z_0 have disjoint spectra it follows that A is block diagonal:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix}.$$

Hence $Z_0 = -A_0^2$. Put

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & A_0 \end{bmatrix};$$

then it follows that $-\tilde{A}^2 = \tilde{X}^{[*]}\tilde{X}$, while $\operatorname{Ker}\tilde{A} = \operatorname{Ker}\tilde{X}$, since $\operatorname{Ker}A = \operatorname{Ker}X$. We conclude that if X admits an H-Hamiltonian polar decomposition, then so does $\tilde{X}$. Conversely, assume that $\tilde{X}$ has an H-Hamiltonian polar decomposition. Then

$$\tilde{X}^{[*]}\tilde{X} = \begin{bmatrix} 0 & 0 \\ 0 & Z_0 \end{bmatrix} = -B^2$$

for some H-Hamiltonian B such that $\operatorname{Ker}B = \operatorname{Ker}\tilde{X}$. The latter property ensures that with respect to the H-orthogonal decomposition (2.4), B has the form

$$B = \begin{bmatrix} 0 & ? \\ 0 & ? \end{bmatrix},$$

and the H-Hamiltonian property of B ensures that $B = O \oplus B_0$ for some B_0. The matrix B_0 is actually such that $H_0B_0 = -B_0^T H_0$ and $\{0\} \oplus \operatorname{Ker}B_0 = \operatorname{Ker}X$. Let B_1 be any H_1-Hamiltonian matrix such that $Z_1 = -B_1^2$, which exists by [1]. Put $A = B_1 \oplus B_0$; then A is H-Hamiltonian, $A^2 = -X^{[*]}X$, and $\operatorname{Ker}A = \operatorname{Ker}X$. Thus, if $\tilde{X}$ admits an H-Hamiltonian polar decomposition, then so does X. We have shown that X admits an H-Hamiltonian polar decomposition if and only if $\tilde{X}$ does, and $\tilde{X}^{[*]}\tilde{X}$ is nilpotent. Moreover, the conditions of Theorem 2.2 are satisfied for X if and only if they are satisfied for $\tilde{X}$. So we may indeed assume in the remainder of the proof that $X^{[*]}X$ is nilpotent.

To prove necessity, assume that $X = UA$ for an H-symplectic U and an H-Hamiltonian A. Bringing the pair (A, H) into canonical form (see Lemma 2.1(b)) and considering separately each orthogonal summand corresponding to the eigenvalue zero of A, we can assume that either

$$A = J_{2p}(0), \qquad H = \varepsilon F_{2p}$$

with respect to some basis $e_1, \ldots, e_{2p}$ or

$$A = \begin{bmatrix} J_{2p-1}(0) & 0 \\ 0 & -J_{2p-1}(0)^T \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & I_{2p-1} \\ -I_{2p-1} & 0 \end{bmatrix}$$

with respect to some basis $e_1, \dots, e_{4p-2}$.

In the former case take the square of A. Consider the vectors

$$f_i = (-1)^{i-1}\frac{1}{\sqrt{2}}(e_{2i-1} + e_{2i})$$

together with the vectors

$$g_i = (-1)^{i-1}\frac{\varepsilon}{\sqrt{2}}(e_{2i-1} - e_{2i})$$

for $i = 1, \dots, p$. Observe that these are real vectors and that $-A^2$ with respect to the basis given by $\{f_1, \dots, f_p; g_p, \dots, g_1\}$ has the form $J_p(0) \oplus J_p(0)^T$. Moreover, the matrix H with respect to this basis has the form

$$\begin{bmatrix} 0 & I_{2p-1} \\ -I_{2p-1} & 0 \end{bmatrix}.$$

Finally, the kernel of A, and hence of X, is given by $\operatorname{Ker} A = \operatorname{span}\{e_1\} = \operatorname{span}\{f_1 + \varepsilon g_1\}$.

In the latter case, take as a basis

$$\begin{array}{ll} e_1, -e_3, \dots, (-1)^{p-1}e_{2p-1}; & e_2, -e_4, \dots, (-1)^p e_{2p-2}; \\ e_{2p}, -e_{2p+2}, \dots, (-1)^{p-1}e_{4p-2}; & e_{2p+1}, -e_{2p+3}, \dots, (-1)^p e_{4p-3}, \end{array}$$

in this order. With respect to this basis $(-A^2, H)$ has the form (2.2), and, moreover, $\operatorname{Ker} A = \operatorname{span}\{e_1, e_{4p-2}\}$, which proves necessity.

To prove sufficiency, we argue as in the proof of Theorem 4.4 in [2]. As observed above, we may assume that X is such that $X^{[*]}X$ is nilpotent. Let us assume that the pair $(X^{[*]}X, H)$ is in the form as described in this theorem with respect to some basis $\{e_{r,j}\}_{r=0,j=1}^{m,\ell_r}$, where $l_0 = 2k_0$, $l_r = 2k_r$ ($r = 1, \dots, m_1$), and $l_r = 4k_r - 2$ ($r = m_1 + 1, \dots, m$). We shall produce for each block (B_r, G_r) a matrix A_r such that $G_r A_r = -A_r^T G_r$ and $-A_r^2 = B_r$, and finally $\operatorname{Ker} A_r = \operatorname{Ker} X \cap \operatorname{span}\{e_{r,j}\}_{j=1}^{\ell_r}$, where $\operatorname{Ker} X$ is given by (2.3).

For the block (B_0, H_0) this is trivial: take $A_0 = B_0 = 0_{2k_0 \times 2k_0}$, the zero matrix of order k_0. Thus we have only to consider the blocks (B_r, H_r) with $r \geq 1$. First consider such a block of type (2.2). Let S be a matrix with the vectors

$$\begin{array}{l} e_{r,1}, e_{r,k_r+1}, -e_{r,2}, -e_{r,k_r+2}, \dots, (-1)^{k_r}e_{r,k_r-1}, (-1)^{k_r}e_{r,2k_r-1}, (-1)^{k_r-1}e_{r,k_r}; \\ e_{r,2k_r}, e_{r,3k_r}, -e_{r,2k_r+1}, e_{r,3k_r+1}, \dots, (-1)^{k_r}e_{r,3k_r-2}, (-1)^{k_r}e_{r,4k_r-2}, (-1)^{k_r-1}e_{r,3k_r-1} \end{array} \tag{2.7}$$

as its columns, in this order. Then

$$S^{-1}B_r S = -\begin{bmatrix} J_{2k_r-1}(0) & 0 \\ 0 & -J_{2k_r-1}(0)^T \end{bmatrix}^2. \quad S^T G_r S = G_r.$$

Put

$$A_r = S\begin{bmatrix} J_{2k_r-1}(0) & 0 \\ 0 & -J_{2k_r-1}(0)^T \end{bmatrix} S^{-1}.$$

Then $-A_r^2 = B_r$ and

$$\operatorname{Ker} A_r = \operatorname{span}\{e_{r,1}, e_{r,4k_r-2}\} = \operatorname{Ker} X \cap \operatorname{span}\{e_{r,j}\}_{j=1}^{\ell_r}.$$

Next, consider a block B_r of type (2.1). Let S be the matrix with the following vectors as its columns:

$$\begin{aligned}\frac{1}{\sqrt{2}}(e_{r,1}+\varepsilon_r e_{r,2k_r}), \frac{1}{\sqrt{2}}(e_{r,1}-\varepsilon_r e_{r,2k_r}), -\frac{1}{\sqrt{2}}(e_{r,2}+\varepsilon_r e_{r,2k_r-1}),\\ -\frac{1}{\sqrt{2}}(e_{r,2}-\varepsilon_r e_{r,2k_r-1}), \dots , (-1)^{k_r-1}\frac{1}{\sqrt{2}}(e_{r,k_r}+\varepsilon_r e_{r,k_r+1}),\\ (-1)^{k_r-1}\frac{1}{\sqrt{2}}(e_{r,k_r}-\varepsilon_r e_{r,k_r+1}).\end{aligned} \tag{2.8}$$

It is assumed that the vectors appear in S in the same order. Then

$$S^{-1}B_rS = -J_{2k_r}(0)^2 \ , \ S^TG_rS = \varepsilon_r F_{2k_r}.$$

Let $A_r = SJ_{2k_r}(0)S^{-1}$. Then $A_r^2 = -B_r$ and

$$\operatorname{Ker} A_r = \operatorname{span}\{e_{r,1}+\varepsilon_r e_{r,2k_r}\} = \operatorname{Ker} X \cap \operatorname{span}\{e_{r,j}\}_{j=1}^{\ell_r},$$

as desired. ◻

The proof of Theorem 2.2 shows that the signs ε_r coincide with the signs ε_j in the canonical form of (A,H) corresponding to the blocks $(J_{2p_j}(0), \varepsilon_j F_{2p_j})$ as in Lemma 2.1(b); here A is the H-Hamiltonian matrix in an H-Hamiltonian polar decomposition $X = UA$ of X.

3. Comparison with existing polar decompositions. Hamiltonian polar decompositions can be compared with the polar decompositions studied in [2, 3, 4]. To do so, note that an H-Hamiltonian polar decomposition $X = UA$ gives rise to the iH-polar decomposition (in the terminology of [2]) $iX = U(iA)$. Here iA is iH-self-adjoint and U is H-symplectic, therefore also iH-unitary. Denoting by $^{[*]}$ the iH-adjoint operation we have $(iX)^{[*]}(iX) = X^{[*]}X$ (note that the definition of $^{[*]}$ given in section 1 coincides with iH-adjoint operation for real matrices:

$$H^{-1}X^TH = (iH)^{-1}X^*(iH)$$

for real X). Since by Theorem 1.2, $X^{[*]}X$ can be put in the form (1.2), it is clear that the partial multiplicities of $X^{[*]}X$ occur only in pairs and that the signs in the iH-sign characteristic of $(iX)^{[*]}(iX) = X^{[*]}X$ corresponding to each pair of multiplicities associated with a real eigenvalue of $(iX)^{[*]}(iX) = X^{[*]}X$ are opposite. Compare the canonical form of H-skew-Hamiltonian matrices; see Lemma 2.1(a).

In what follows we shall denote by Q_k the $k \times k$ matrix with zeros everywhere except on the south-west/north-east diagonal, where there are ones.

Combining the observation above with the necessary and sufficient conditions (obtained in [2]) for the existence of an iH-polar decomposition of iX, we obtain the following result.

THEOREM 3.1. *Let X be a real $2n \times 2n$ matrix and H a real skew-symmetric invertible $2n \times 2n$ matrix. Then there exists an iH-polar decomposition of iX if and only if the part of the canonical form of $(X^{[*]}X, iH)$ corresponding to the zero eigenvalue of $X^{[*]}X$ can be represented in the form*

$$(\operatorname{diag}(B_j)_{j=0}^m, \operatorname{diag}(G_j)_{j=0}^m),$$

where

(i) *B_0 is the zero matrix of order $2k_0$ and $G_0 = I_{k_0} \oplus -I_{k_0}$,*

(ii) *$m = m_1 + m_2$, and for each $j = 1, \dots, m_1$ we have*

$$B_j = \begin{bmatrix} J_{k_j}(0) & 0 \\ 0 & J_{k_j}(0) \end{bmatrix}, \qquad G_j = \begin{bmatrix} Q_{k_j} & 0 \\ 0 & -Q_{k_j} \end{bmatrix}, \tag{3.1}$$

while for $j = m_1 + 1, \dots, m_1 + m_2$ we have

$$B_j = \begin{bmatrix} J_{k_j}(0) & 0 & 0 & 0 \\ 0 & J_{k_j-1}(0) & 0 & 0 \\ 0 & 0 & J_{k_j}(0) & 0 \\ 0 & 0 & 0 & J_{k_j-1}(0) \end{bmatrix},$$

$$G_j = \begin{bmatrix} Q_{k_j} & 0 & 0 & 0 \\ 0 & Q_{k_j-1} & 0 & 0 \\ 0 & 0 & -Q_{k_j} & 0 \\ 0 & 0 & 0 & -Q_{k_j-1} \end{bmatrix}, \tag{3.2}$$

(iii) *and, denoting the corresponding basis in* $\operatorname{Ker}(X^{[*]}X)^{2n} \subseteq \mathbb{C}^{2n}$ *in which the form* (i), (3.1), (3.2) *is achieved, by $\{e_{r,j}\}_{r=0,j=1}^{m,\ell_r}$, where $\ell_0 = 2k_0$, $\ell_r = 2k_r$ for $r = 1, \dots, m_1$, and $\ell_r = 4k_r - 2$ for $r = m_1 + 1, \dots, m_2$ we have*

$$\begin{aligned} \operatorname{Ker} X = {} & \operatorname{span}\{e_{r,1} + e_{r,k_r+1} \mid r = 1, \dots, m_1\} \\ & + \operatorname{span}\{e_{r,1}, e_{r,2k_r} \mid r = m_1 + 1, \dots, m_2\} \\ & + \operatorname{span}\{e_{0,j}\}_{j=1}^{2k_0}. \end{aligned}$$

The clarifications made after Theorem 2.2 apply to Theorem 3.1 as well.

It is easily verified that the conditions of Theorem 3.1 are necessary for the existence of an H-Hamiltonian polar decomposition of X. Theorem 2.2 shows that they are also sufficient. We indicate (omitting many details) how one can derive Theorem 2.2 directly from Theorem 3.1. First write $(X^{[*]}X, iH)$ in canonical form as in Theorem 2.1 of [2] for $F = \mathbb{C}$ and take the complex conjugate of (2.2) and (2.3) of [2]. This leads to real Jordan blocks with opposite signs, and hence they can be arranged in pairs having opposite signs. Further, the condition in Theorem 4.4 of [2] on the negative eigenvalues of $X^{[*]}X$ to guarantee the existence of an (iH)-polar decomposition of X turns out to be superfluous. Next, writing $(X^{[*]}X, iH)$ in canonical form as in Theorem 2.1 of [2] for $F = \mathbb{C}$ with consecutive real Jordan blocks of equal size and opposite sign and letting $\sigma_1^j, \dots, \sigma_{k_j}^j$ $(j = 1, \dots, \alpha)$ stand for the first $k_1 + \cdots + k_\alpha$ columns of the complex matrix S that transforms $(X^{[*]}X, iH)$ to the canonical form, we obtain the part of the canonical form of $(X^{[*]}X, H)$ according to Lemma 2.1(a) if we let the columns of the new S be the vectors

$$\begin{gathered} \rho_j^1 - \varepsilon_j \tau_{k_j}^j, \rho_j^2 - \varepsilon_j \tau_{k_j-1}^j, \dots, \rho_j^{k_j} - \varepsilon_j \tau_1^j, \\ \rho_j^1 + \varepsilon_j \tau_{k_j}^j, \rho_j^2 + \varepsilon_j \tau_{k_j-1}^j, \dots, \rho_j^{k_j} + \varepsilon_j \tau_1^j, \end{gathered}$$

where $j = 1, \dots, \alpha$ and ρ_r^j and τ_r^j are the real and imaginary parts of σ_r^j, and we arrive at a direct derivation of Theorem 2.2 from Theorem 3.1. We have omitted in the above discussion consideration of Jordan blocks corresponding to nonreal eigenvalues of $X^{[*]}X$.

COROLLARY 3.2. *A real matrix X has an H-Hamiltonian polar decomposition with respect to a real invertible skew-symmetric matrix H if and only if iX has an iH-polar decomposition (over the field of complex numbers).*

REFERENCES

[1] H. Fassbender, D. S. Mackey, N. Mackey, and H. Xu, *Hamiltonian square roots of skew-Hamiltonian matrices*, Linear Algebra Appl., 287 (1999), pp. 125–159.

[2] Yu. Bolshakov, C. V. M. van der Mee, A. C. M. Ran, B. Reichstein, and L. Rodman, *Polar decompositions in finite dimensional indefinite scalar product spaces: General theory*, Linear Algebra Appl., 261 (1997), pp. 91–141.

[3] Yu. Bolshakov, C. V. M. van der Mee, A. C. M. Ran, B. Reichstein, and L. Rodman, *Extensions of isometries in finite-dimensional indefinite scalar product spaces and polar decompositions*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 752–774.

[4] Yu. Bolshakov, C. V. M. van der Mee, A. C. M. Ran, B. Reichstein, and L. Rodman, *Polar decompositions in finite dimensional indefinite scalar product spaces: Special cases and applications*, in Recent Developments in Operator Theory and Its Applications, Oper. Theory Adv. Appl. 87, I. Gohberg, P. Lancaster, and P. N. Shivakumar, eds., Birkhäuser, Basel, 1996, pp. 61–94. Erratum in Integral Equations Operator Theory, 27 (1997), pp. 497–501.

[5] D. Ž. Djoković, J. Patera, P. Winternitz, and H. Zassenhaus, *Normal forms of elements of classical real and complex Lie and Jordan algebras*, J. Math. Phys., 24 (1983), pp. 1363–1374.

[6] C. V. M. van der Mee, A. C. M. Ran, and L. Rodman, *Stability of self-adjoint square roots and polar decompositions in indefinite scalar product spaces*, Linear Algebra Appl., 302/303 (1999), pp. 77–104.

[7] R. C. Thompson, *Pencils of complex and real symmetric and skew matrices*, Linear Algebra Appl., 147 (1991), pp. 323–371.