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Stability of self-adjoint square roots and polar decompositions in indefinite scalar product spaces

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Abstract

Continuity properties of factors in polar decompositions of matrices with respect to indefinite scalar products are studied. The matrix having the polar decomposition and the indefinite scalar product are allowed to vary. Closely related properties of a self-adjoint (with respect to an indefinite scalar product) perturbed matrix to have a self-adjoint square root, or to have a representation of the form X^*X, are also studied. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let F be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Choose a fixed real symmetric (if $F = \mathbb{R}$) or complex Hermitian (if $F = \mathbb{C}$) invertible $n \times n$ matrix H. Consider the scalar product induced by H by the formula $[x, y] = \langle Hx, y\rangle$, $x, y \in F^n$. Here $\langle\cdot\,,\,\cdot\rangle$ stands for the usual scalar product in F^n, i.e.,

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j \overline{y_j},$$

where x and y are the column vectors with components $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, respectively, and $\overline{y_j} = y_j$ if $F = \mathbb{R}$. The scalar product $[\cdot\,,\,\cdot]$ is non-degenerate ($[x, y] = 0$ for all $y \in F^n$ implies $x = 0$), but is indefinite in general. The vector $x \in F^n$ is called *positive* if $[x, x] > 0$, *neutral* if $[x, x] = 0$ and *negative* if $[x, x] < 0$.

Well-known concepts related to scalar products are defined in an obvious way. Thus, given an $n \times n$ matrix A over F, the H-adjoint A^H is defined by $[Ax, y] = [x, A^H y]$ for all $x, y \in F^n$. In that case $A^H = H^{-1}A^*H$, where A^* denotes the conjugate transpose of A (with $A^* = A^{\mathrm{T}}$, the transpose of A, if $F = \mathbb{R}$). An $n \times n$ matrix A is called *H-self-adjoint* if $A^H = A$ (or equivalently, if HA is Hermitian). An $n \times n$ matrix U is called *H-unitary* if $[Ux, Uy] = [x, y]$ for all $x, y \in F^n$ (or equivalently, if $U^*HU = H$).

In a number of recent papers [8,5–7] decompositions of an $n \times n$ matrix X over F of the form

$$X = UA, \tag{1.1}$$

where U is H-unitary and A is H-self-adjoint, have been studied. By analogy with the standard polar decomposition $X = UA$, where U is unitary and A is positive semidefinite, decompositions (1.1) are called *H-polar decompositions* of X. In particular, necessary and sufficient conditions on a matrix X to have an H-polar decomposition in various equivalent forms have been established in [5] and further specialized to the particular case where H has exactly one positive eigenvalue in [6]. For H-contractions (i.e., matrices X for which $H - HX^HX$ is semidefinite self-adjoint) and H-plus matrices (i.e., matrices X for which there exists $\mu \geqslant 0$ such that $[X^HXu, u] \geqslant \mu[u, u]$ for every $u \in F^n$) these results are special cases of results known for Krein spaces [11,12,18,19]. Essentially, to prove the existence of and to actually construct an H-polar decomposition of a given $n \times n$ matrix X, one needs to find an H-self-adjoint matrix A such that

$$\begin{aligned} X^H X &= A^2 \\ \operatorname{Ker} X &= \operatorname{Ker} A, \end{aligned} \tag{1.2}$$

where Ker B stands for the null space of a matrix B [8,5]. Once A is known, the map $Au \mapsto Xu$ is an H-isometry from the range Im A of A onto the range Im X of X, which can be extended to an H-unitary matrix U as a result of Witt's theorem [1,7].

In this paper, we study stability properties of H-polar decompositions of a given $n \times n$ matrix X, more precisely, the local problem of specifying those $n \times n$ matrices X over F having an H-polar decomposition where the factors in (1.1) can be chosen to depend continuously on X under small perturbations of X. Our main results on stability of H-polar decompositions are given in Sections 3 and 4. It turns out that, for the case $F = \mathbb{C}$, there exist stable H-polar decompositions of X if and only if $\sigma(X^H X) \cap (-\infty, 0] = \emptyset$. In the real case, this condition is only sufficient but not necessary, a phenomenon that appears already in the Hilbert space situation (see [15]). Nevertheless, for $F = \mathbb{R}$ we give necessary and sufficient conditions for the existence of stable H-polar decompositions of X.

In connection with H-polar decompositions, several other classes of matrices relative to an indefinite scalar product appear naturally, namely, those matrices that can be written in the form $X^H X$, and those for which there exists an H-self-adjoint square root. We study these classes in Sections 2 and 3. In particular, we characterize those $n \times n$ matrices Y over F which have an H-self-adjoint square root A (i.e., $A^H = A$ and $A^2 = Y$) that depends continuously on Y under small perturbations of Y.

The following notations will be used. The block diagonal matrix with matrices $Z_1, \ldots, Z_k$ on the diagonal is denoted by $Z_1 \oplus \cdots \oplus Z_k$. The set of eigenvalues (including the nonreal eigenvalues for real matrices) of a matrix X is denoted by $\sigma(X)$. The norm $\|A\|$ of a matrix A is the operator norm (the largest singular value). We denote by $i_+(Y)$ (resp. $i_-(Y)$) the number of positive (resp. negative) eigenvalues of a Hermitian matrix Y, multiplicities taken into account.

Unless indicated otherwise, the results are valid for both the real and the complex case.

Throughout the paper it will be assumed that the indefinite scalar products involved are genuinely indefinite, i.e., there exist vectors x and y for which $[x,x] < 0 < [y,y]$. The problem concerning stability of polar decomposition with respect to a definite scalar product has been addressed in [15]; see also ch. VII of [3].

We conclude the introduction by recalling the well-known canonical form for pairs (A, H), where A is H-self-adjoint. Let $J_k(\lambda)$ denote the $k \times k$ upper triangular Jordan block with $\lambda \in \mathbb{C}$ on the diagonal and let $J_k(\lambda \pm \mathrm{i}\mu)$ denote the matrix

$$J_k(\lambda \pm \mathrm{i}\mu) = \begin{bmatrix} \lambda & \mu & 1 & 0 & & & & & & \\ -\mu & \lambda & 0 & 1 & & & & 0 & & \\ 0 & 0 & \lambda & \mu & 1 & 0 & & & & \\ 0 & 0 & -\mu & \lambda & 0 & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & & 1 & 0 \\ & & & & & & & & 0 & 1 \\ & & 0 & & & & 0 & 0 & \lambda & \mu \\ & & & & & & 0 & 0 & -\mu & \lambda \end{bmatrix}, \tag{1.3}$$

where $\lambda, \mu \in \mathbb{R}$, $\mu > 0$ and k is necessarily even. Note that although we define two different J_k's, it will always be clear from the context which one is meant. We denote by Q_k the $k \times k$ matrix with ones on the south-west north-east diagonal and zeros elsewhere. Then the following characterization of pairs (A, H) goes back to Weierstrass and Kronecker (see, for example, [13,20] for a complete proof).

Theorem 1.1. *Let H be an invertible Hermitian $n \times n$ matrix (over F), and let $A \in F^{n\times n}$ be H-self-adjoint. Then there exists an invertible S over F such that $S^{-1}AS$ and S^*HS have the form*

$$S^{-1}AS = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus \left[J_{k_{\alpha+1}}(\lambda_{\alpha+1}) \oplus J_{k_{\alpha+1}}(\bar{\lambda}_{\alpha+1})\right] \oplus \cdots \oplus \left[J_{k_\beta}(\lambda_\beta) \oplus J_{k_\beta}(\bar{\lambda}_\beta)\right] \tag{1.4}$$

if $F = \mathbb{C}$, where $\lambda_1, \ldots, \lambda_\alpha$ are real and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ are nonreal with positive imaginary parts;

$$S^{-1}AS = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus J_{2k_{\alpha+1}}(\lambda_{\alpha+1} \pm \mathrm{i}\mu_{\alpha+1}) \oplus \cdots \oplus J_{2k_\beta}(\lambda_\beta \pm \mathrm{i}\mu_\beta), \tag{1.5}$$

if $F = \mathbb{R}$, where $\lambda_1, \ldots, \lambda_\beta$ are real and $\mu_{\alpha+1}, \ldots, \mu_\beta$ are positive;

$$S^*HS = \varepsilon_1 Q_{k_1} \oplus \cdots \oplus \varepsilon_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta} \tag{1.6}$$

for both cases ($F = \mathbb{R}$ or $F = \mathbb{C}$), where the signs $\varepsilon_1, \ldots, \varepsilon_\alpha$ are ± 1. For a given pair (A, H), where A is H-self-adjoint, the canonical form (1.4) *and* (1.6) (*for $F = \mathbb{C}$*) *or* (1.5) *and* (1.6) (*for $F = \mathbb{R}$*) *is unique up to permutation of H-orthogonal components in* (1.6), *and the same simultaneous permutation of the corresponding blocks in* (1.4) *or* (1.5), *as the case may be.*

2. Matrices of the form $X^H X$ and their stability

We start with a description of matrices of the form $X^H X$, where $H \in F^{n\times n}$ is an invertible Hermitian matrix. We consider the case where the indefinite scalar product is fixed, as well as the case where it is allowed to vary. Our first result is the following theorem.

Theorem 2.1. *Let $A \in F^{n\times n}$. Then $A = X^H X$ for some $X \in F^{n\times n}$ and some indefinite scalar product $[\cdot,\cdot] = \langle H\cdot,\cdot\rangle$ in F^n if and only if the following two conditions are satisfied:*
(i) *A is similar to a real matrix;*
(ii) $\det A \geqslant 0$.

Of course, the condition (i) is trivial if $F = \mathbb{R}$. Furthermore, since the eigenvalues of a real matrix are symmetric relative to the real axis and complex conjugate eigenvalues λ and $\bar{\lambda}$ have the same multiplicities, the condition (ii) in Theorem 2.1 may be replaced by the following:

(iii) *Either A is singular, or A is nonsingular and the sum of the algebraic multiplicities corresponding to its negative eigenvalues (if any) is even.*

Proof. The necessity of the conditions (i) and (ii) is easy to see. Indeed, if $A = X^H X = H^{-1}X^*HX$, then obviously $\det A = (\det X^*)(\det X) = |\det X|^2 \geqslant 0$. Moreover, $A^* = X^*HXH^{-1} = HAH^{-1}$. In particular, A is similar to A^*; this condition is well known and easily seen (using the real Jordan form) to be equivalent to (i).

For the converse, we start with a well-known fact. Let $H_1, H_2 \in F^{n\times n}$ be Hermitian matrices. Then there exists $X \in F^{n\times n}$ such that $H_1 = X^*H_2X$ if and only if the following two inequalities hold:

$$i_+(H_1) \leqslant i_+(H_2), \qquad i_-(H_1) \leqslant i_-(H_2). \tag{2.1}$$

Indeed, if (2.1) holds, then the existence of X is easily seen upon reducing H_1 and H_2 to diagonal form with 1's, -1's and 0's on the diagonal, via congruence: $H_1 \to Z_1^*H_1Z_1$, $H_2 \to Z_2^*H_2Z_2$, for suitable invertible Z_1 and Z_2. Conversely, let $X \in F^{n\times n}$ be such that $H_1 = X^*H_2X$. Writing $X = Z_1DZ_2$ for suitable invertible matrices Z_1 and Z_2, where D is a diagonal matrix with 1's and 0's on the diagonal, and replacing H_1 with $(Z_2^*)^{-1}H_1Z_2^{-1}$ and H_2 with $Z_1^*H_2Z_1$, we can assume without loss of generality that $X = D$. But then (2.1) follows from the interlacing inequalities between the eigenvalues of a Hermitian matrix and the eigenvalues of its principal submatrices.

Assume that A satisfies (i) and (iii). It is well known (see, e.g., Corollary 3.5 of [8]) that (i) is equivalent to A being H-self-adjoint for some invertible

Hermitian matrix H. Note that the problem is invariant under simultaneous similarity of A and congruence of H; in other words, if there is an $X \in F^{n\times n}$ such that $A = X^H X$, then for any invertible $S \in F^{n\times n}$ there is a $Y \in F^{n\times n}$ such that $S^{-1}AS = Y^{S^*HS}Y$ (in fact, $Y = S^{-1}XS$ will do), and vice versa. Therefore, without loss of generality we may assume that A and H are given by the canonical form of Theorem 1.1.

It is known (see Theorem 4.4 in [5], or Theorem 3.1 in the next section for more details) that if A_0 is H_0-self-adjoint and has no negative or zero eigenvalues, then $A_0 = X_0^{H_0}X_0$ for some X_0; moreover, such X_0 can also be chosen to be H_0-self-adjoint. Thus, if the matrix A given by (1.4) or by (1.5) has nonreal or positive eigenvalues, we can complete the proof using this observation and induction on the size of the matrices A and H.

It remains therefore to consider the case when

$$A = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_p}(\lambda_p), \qquad H = \varepsilon_1 Q_{k_1} \oplus \cdots \oplus \varepsilon_p Q_{k_p}, \tag{2.2}$$

where $\lambda_1, \ldots, \lambda_p$ are nonpositive real numbers. Say, $\lambda_1, \ldots, \lambda_r < 0 = \lambda_{r+1} = \cdots = \lambda_p$. A straightforward inspection shows that the integer $i_+((\varepsilon_j Q_{k_j})J_{k_j}(\lambda_j)) - i_+(\varepsilon_j Q_{k_j})$ is given by the following table:

$$\begin{array}{rl} 0 & \text{if } k_j \text{ is even and } j \leqslant r; \\ 1 & \text{if } k_j \text{ is odd}, \ \varepsilon_j = -1, \ \text{and } j \leqslant r; \\ -1 & \text{if } k_j \text{ is odd}, \ \varepsilon_j = 1, \ \text{and } j \leqslant r; \\ 0 & \text{if } k_j \text{ is even}, \ j > r, \ \text{and } \varepsilon_j = 1; \\ -1 & \text{if } k_j \text{ is odd}, \ j > r, \ \text{and } \varepsilon_j = 1; \\ 0 & \text{if } k_j \text{ is odd}, \ j > r, \ \text{and } \varepsilon_j = -1; \\ -1 & \text{if } k_j \text{ is even}, \ j > r, \ \text{and } \varepsilon_j = -1. \end{array} \tag{2.3}$$

In what follows, we denote by $\#K$ the cardinality of a finite set K. Therefore,

$$\begin{aligned} i_+(HA) - i_+(H) = {} & \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = -1, \ j \leqslant r\} \\ & - \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = 1, \ j \leqslant r\} \\ & - \#\{j : \ \varepsilon_j(-1)^{k_j} = -1, \ j > r\}. \end{aligned}$$

Analogously,

$$\begin{aligned} i_-(HA) - i_-(H) = {} & \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = 1, \ j \leqslant r\} \\ & - \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = -1, \ j \leqslant r\} \\ & - \#\{j : \ \varepsilon_j(-1)^{k_j} = 1, \ j > r\}. \end{aligned}$$

Let

$$q = \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = -1, \ j \leqslant r\} - \#\{j : \ k_j \text{ odd}, \ \varepsilon_j = 1, \ j \leqslant r\}.$$

We obtain both of the inequalities

$$i_+(HA) \leqslant i_+(H), \qquad i_-(HA) \leqslant i_-(H) \tag{2.4}$$

if and only if

$$-\#\{j:\ \varepsilon_j(-1)^{k_j} = 1,\ j > r\} \leqslant q \leqslant \#\{j:\ \varepsilon_j(-1)^{k_j} = -1,\ j > r\}. \tag{2.5}$$

If $p = r$, then condition (iii) guarantees that the signs ε_j can be assigned in such a way that $q = 0$, so that (2.5) holds. If $p > r$ and $\sum_{j=1}^{r} k_j$ is even then $\#\{j \leqslant r : k_j \text{ is odd}\}$ is even and then we can take $q = 0$ by assigning the signs ε_j in a suitable way. If $p > r$ and $\sum_{j=1}^{r} k_j$ is odd then we can take the signs ε_j so that $q = -1$. Take ε_{r+1} so that $\varepsilon_{r+1}(-1)^{k_{r+1}} = 1$. This can always be done and guarantees that (2.5) will hold. But then also (2.4) holds; by (2.1), there exists $X \in F^{n\times n}$ such that $HA = X^*HX$, i.e., $A = X^H X$.

This completes the proof of Theorem 2.1. □

If the indefinite scalar product is kept fixed in (1.1), then we obtain a related (but different) problem: Given an invertible Hermitian H, identify those H-self-adjoint matrices A that can be represented in the form $A = X^H X$ for some X. This problem has been addressed in the literature (see, e.g., Section VII.2 in [4], where a solution of this problem is given in the context of infinite dimensional spaces with indefinite scalar products and the inequalities (2.4) appear). In fact, using (2.1), it is immediate that $A \in F^{n\times n}$ can be written in the form $A = X^H X$ for some $X \in F^{n\times n}$ if and only if the inequalities (2.4) hold. This observation was first made in [18]; a parametrization of the set of solutions of $A = X^H X$ (with fixed H) is also given in [18]. On the other hand, by following the arguments of the proof of Theorem 2.1, we obtain:

Theorem 2.2. *Let $A \in F^{n\times n}$ be H-self-adjoint, and let* (2.2), *where $\lambda_j < 0$ if $j \leqslant r$ and $\lambda_j = 0$ if $j > r$, be the part of the canonical form of (A, H) that corresponds to the real nonpositive eigenvalues of A. Then there exists an $X \in F^{n\times n}$ such that $A = X^H X$, if and only if the condition*

$$\begin{aligned}-\#\{j:\ \varepsilon_j(-1)^{k_j} = 1,\ j > r\} &\leqslant \#\{j:\ k_j \text{ odd},\ \varepsilon_j = -1,\ j \leqslant r\} \\ &\quad - \#\{j:\ k_j \text{ odd},\ \varepsilon_j = 1,\ j \leqslant r\} \\ &\leqslant \#\{j:\ \varepsilon_j(-1)^{k_j} = -1,\ j > r\}\end{aligned} \tag{2.6}$$

holds.

Using table (2.3), it is not difficult to see that (2.6) is equivalent to (2.4).

Corollary 2.3. *Let $A \in F^{n\times n}$ be H-self-adjoint with* $\det A \geqslant 0$, *and let α be the number of odd multiplicities* ($=$ *the sizes of Jordan blocks*) *corresponding to the negative eigenvalues of A. Then there exists an invertible Hermitian matrix $H_0 \in F^{n\times n}$ such that*

$$|i_+(H_0) - i_+(H)| \leqslant \lceil \alpha/2 \rceil,$$

and $A = X^{H_0}X$ *for some* X. *Here* $\lceil m \rceil$ *is the smallest integer greater than or equal to* m.

Indeed, the formula (2.6) shows that, starting with the sequence of signs $(\varepsilon'_1, \ldots, \varepsilon'_p)$ in the part of the canonical form of (A, H) corresponding to the real nonpositive eigenvalues, one needs to replace at most $\lceil \alpha/2 \rceil$ of the signs $(\varepsilon'_1, \ldots, \varepsilon'_p)$ with their opposites to satisfy the inequalities (2.6). Each such replacement changes the number of positive eigenvalues of H by at most one.

Next, we describe matrices having the form $X^H X$ under small perturbations.

Theorem 2.4. (a) *Assume that* $A \in F^{n\times n}$ *is nonsingular and has a representation* $A = X^H X$ *for some* $X \in F^{n\times n}$ *and some indefinite scalar product* $[\cdot,\cdot] = \langle H\cdot,\cdot\rangle$. *Then there exists* $\varepsilon > 0$ *such that every matrix* $B \in F^{n\times n}$ *has such a representation provided* $\|B - A\| < \varepsilon$ *and* B *is similar to a real matrix.*

(b) *Assume* A *is as in the part* (a). *Then there exists* $\varepsilon > 0$ *such that every matrix* $B \in F^{n\times n}$ *has a representation* $B = Y^G Y$ *for a suitable* $Y \in F^{n\times n}$, *provided* $\|B - A\| + \|G - H\| < \varepsilon$ *and* B *is* G*-self-adjoint. Moreover, such a* Y *can be chosen so that*

$$\|Y - X\| \leqslant K(\|B - A\| + \|G - H\|), \tag{2.7}$$

where the constant $K > 0$ *depends on* A, X *and* H *only.*

(c) *Conversely, assume that* $A \in F^{n\times n}$ *is* H*-self-adjoint and singular. Then for every* $\varepsilon > 0$ *there exists an* H*-self-adjoint matrix* B *such that* $\|A - B\| < \varepsilon$ *and* B *does not admit a representation of the form* $Y^G Y$ *for any invertible Hermitian matrix* $G \in F^{n\times n}$.

Proof. Part (a) is immediate from Theorem 2.1, taking into account that $\det A > 0$ and therefore $\det B > 0$ for all nearby matrices B that are similar to a real matrix.

For part (b) observe that the inequalities (2.4) hold true. But A is nonsingular, so in fact (2.4) are valid with the equality sign. Since the integers $i_\pm(Z)$ remain constant for all Hermitian matrices Z sufficiently close to a given nonsingular Hermitian matrix, we obtain $i_\pm(GB) = i_\pm(G)$ for all Hermitian G sufficiently close to H, and all matrices B sufficiently close to A provided B is G-self-adjoint. Then B will have the desired form $B = Y^G Y$. To obtain the inequality (2.7), observe that $GB = Y^* GY$ is close to $HA = X^* HX$, and therefore (for ε small enough) $X^* HX$ and $Y^* GY$ have the same inertia. Now use the result of [17], according to which Y can be chosen close to X, with $\|Y - X\|$ of the same order of magnitude as $\|GB - HA\|$.

For part (c), by Theorem 2.1, we need to exhibit H-self-adjoint matrices B with negative determinant that are arbitrarily close to A. Without loss of generality we can assume that the pair (A, H) is given by (1.4) and (1.6) if $F = \mathbb{C}$, and by (1.5) and (1.6) if $F = \mathbb{R}$. Taking blocks together, write

$$A = A_0 \oplus J_{m_1}(0) \oplus \cdots \oplus J_{m_r}(0), \;\; H = H_0 \oplus \delta_1 Q_{m_1} \oplus \cdots \oplus \delta_r Q_{m_r},$$

where A_0 is invertible ($\delta_j = \pm 1$). Denoting by K_m the $m \times m$ matrix with 1 in the lower left corner and 0 everywhere else, we let

$$B = A_0 \oplus (J_{m_1}(0) + \alpha_1 K_{m_1}) \oplus \cdots \oplus (J_{m_r}(0) + \alpha_r K_{m_r}),$$

where $\alpha_1, \ldots, \alpha_r$ are nonzero real numbers. It is straightforward to check that B is H-self-adjoint. Moreover,

$$\det B = (\det A_0)(-1)^{\Sigma(m_j - 1)} \alpha_1 \ldots \alpha_r.$$

So, by choosing α_js having suitable signs, we make $\det B$ negative, and by choosing α_j's sufficiently close to zero, we can make B as close to A as we wish. $\square$

3. Matrices having self-adjoint square roots

In this section, we characterize stability of matrices having an H-self-adjoint square root. Clearly, if $A \in F^{n\times n}$ has an H-self-adjoint square root, then necessarily A is of the form $A = X^H X$ for some H. However, the converse is generally not true, as follows from a description of matrices having an H-self-adjoint square root in terms of the canonical form of Theorem 1.1:

Theorem 3.1. *Let Y be an H-self-adjoint $n \times n$ matrix over F. Then there exists an H-self-adjoint matrix A such that $A^2 = Y$ if and only if the canonical form of (Y, H) has the following properties*:

(i) *The part of the canonical form of (Y, H) pertaining to each eigenvalue $-\beta^2$ with $\beta > 0$ has the form $(\oplus_{j=1}^r (J_{k_j}(-\beta^2) \oplus J_{k_j}(-\beta^2)), \oplus_{j=1}^r (Q_{k_j} \oplus (-Q_{k_j})))$, where $k_1 \leqslant \cdots \leqslant k_r$.*

(ii) *The part of the canonical form of (Y, H) pertaining to the zero eigenvalue can be written as $\left(J^{(1)} \oplus J^{(2)} \oplus J^{(3)}, Q^{(1)} \oplus Q^{(2)} \oplus Q^{(3)}\right)$, where*

$$J^{(1)} = \bigoplus_{i=1}^{s} (J_{l_i}(0) \oplus J_{l_i}(0)), \quad J^{(2)} = \bigoplus_{j=1}^{t} (J_{m_j}(0) \oplus J_{m_j - 1}(0)),$$

$$J^{(3)} = \bigoplus_{h=1}^{u} J_1(0), \quad Q^{(1)} = \bigoplus_{i=1}^{s} (Q_{l_i} \oplus (-Q_{l_i})),$$

$$Q^{(2)} = \bigoplus_{j=1}^{t} (\varepsilon_{m_j} Q_{m_j} \oplus \varepsilon_{m_j} Q_{m_j - 1}), \quad Q^{(3)} = \bigoplus_{h=1}^{u} \varepsilon_h, \tag{3.1}$$

where $1 \leqslant l_1 \leqslant \cdots \leqslant l_s$, $2 \leqslant m_1 \leqslant \cdots \leqslant m_t$ and $\varepsilon_h, \varepsilon_{m_j} \in \{-1, 1\}$ ($j = 1, \cdots, t$).

Moreover, the corresponding parts of the canonical forms of (Y,H) *and* (A,H) *are related as follows*:

(a) *Let the canonical form of the pair* (Y,H) *contain* $(J_k(\alpha+\mathrm{i}\beta)\oplus J_k(\alpha-\mathrm{i}\beta),Q_{2k})$, *where* $\alpha,\ \beta\in\mathbb{R},\ \beta\neq 0$, *and let* λ *be a complex number such that* $\lambda^2=\alpha+\mathrm{i}\beta$. *Then the canonical form of the pair* (A,H) *contains either* $(J_k(\lambda)\oplus J_k(\bar{\lambda}),Q_{2k})$ *or* $(J_k(-\lambda)\oplus J_k(\overline{-\lambda}),Q_{2k})$.

(b) *If the canonical form of the pair* (Y,H) *contains* $(J_k(-\beta^2)\oplus J_k(-\beta^2),Q_k\oplus(-Q_k))$, *where* $\beta>0$, *then the canonical form of the pair* (A,H) *contains* $(J_k(\mathrm{i}\beta)\oplus J_k(-\mathrm{i}\beta),Q_{2k})$.

(c) *Let the canonical form of the pair* (Y,H) *contain* $(J_k(\mu^2),\varepsilon Q_k)$, *where* $\mu>0$. *Then the canonical form of the pair* (A,H) *contains either* $(J_k(\mu),\varepsilon Q_k)$ *or* $(J_k(-\mu),(-1)^{k+1}\varepsilon Q_k)$.

(d) *If the canonical form of* (Y,H) *contains* $(J_1(0),\varepsilon)$, *which is a part of* $(J^{(3)},Q^{(3)})$, *then the canonical form of* (A,H) *contains* $(J_1(0),\varepsilon)$.

(e) *Let the canonical form of the pair* (Y,H) *contain* $(J_k(0)\oplus J_{k-1}(0),\varepsilon Q_k\oplus\varepsilon Q_{k-1})$, *which is a part of* $(J^{(2)},Q^{(2)})$. *Then the canonical form of the pair* (A,H) *contains* $(J_{2k-1}(0),\varepsilon Q_{2k-1})$. *Moreover, a canonical basis can be chosen in such a way that the eigenvector of A coincides with the eigenvector of the* $k\times k$ *Jordan block of Y.*

(f) *Let the canonical form of the pair* (Y,H) *contain* $(J_k(0)\oplus J_k(0),Q_k\oplus(-Q_k))$, *which is a part of* $(J^{(1)},Q^{(1)})$. *Then the canonical form of the pair* (A,H) *contains either* $(J_{2k}(0),Q_{2k})$ *or* $(J_{2k}(0),-Q_{2k})$. *Moreover, a canonical basis can be chosen in such a way that the eigenvector of A coincides with the sum of the eigenvectors of the two Jordan blocks of Y.*

Observe that in general there may be several possible ways to decompose the part of the canonical form of (Y,H) pertaining to the zero eigenvalue into

$$\left(J^{(1)}\oplus J^{(2)}\oplus J^{(3)},Q^{(1)}\oplus Q^{(2)}\oplus Q^{(3)}\right)$$

with the properties described in (3.1). For every such partition, the *H*-selfadjoint square root *A* of *Y* is described in (a)–(f). The result of Theorem 3.1 is proved in [5] (Lemmas 7.7 and 7.8). We remark that the existence of square roots, without any recourse to the scalar product involved, has been characterized in [10] for complex matrices and in [9] for real matrices.

The following corollary of Theorem 3.1 is easily obtained:

Corollary 3.2. *A matrix* $Y\in F^{n\times n}$ *can be represented in the form* $Y=X^2$ *for some* $X=X^H\in F^{n\times n}$ *and some indefinite scalar product* $[\cdot,\cdot]=\langle H\cdot,\cdot\rangle$ *if and only if the following conditions are satisfied*:

(i) *Y is similar to a real matrix.*

(ii) *For every negative eigenvalue* λ *of Y* (*if any*), *and for every positive integer k*, *the number of partial multiplicities of Y corresponding to the eigenvalue* λ *and equal to k*, *is even* (*possibly zero*).

(iii) *If Y is singular, let* $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_q \geqslant 1$ *be the partial multiplicities of Y corresponding to the zero eigenvalue, arranged in the nonincreasing order. Then the inequalities* $\alpha_{2m-1} - \alpha_{2m} \leqslant 1$ *hold for* $m = 1, 2, \ldots, q/2$ (*if q is odd, we formally put* $\alpha_{q+1} = 0$ *and apply the preceding inequalities for* $m \leqslant (q+1)/2$).

It is instructive to compare Corollary 3.2 and Theorem 2.1. It follows that there are many situations in which A admits a representation $A = X^H X$ for some $X \in F^{n\times n}$, but there is no G-self-adjoint square root of A for any Hermitian invertible matrix G. For example, this is always the case if all the algebraic multiplicities corresponding to the negative eigenvalues of A are even, and for at least one negative eigenvalue of A the number of partial multiplicities corresponding to that eigenvalue is odd.

Next, we describe the stability of matrices having H-self-adjoint square roots. It turns out that this property is stable precisely when the matrix is nonsingular.

Theorem 3.3. *Let Y be an H-self-adjoint matrix such that* $\sigma(Y) \cap (-\infty, 0] = \emptyset$. *Then there exist an H-self-adjoint matrix A satisfying* $A^2 = Y$ *and constants* $\delta, M > 0$, *depending on A, Y and H only, such that for any G-self-adjoint matrix Z with* $\|Y - Z\| < \delta$ *there exists a G-self-adjoint matrix B satisfying* $B^2 = Z$ *such that*

$$\|A - B\| \leqslant M\|Y - Z\|. \tag{3.2}$$

Conversely, let Y be an it H-self-adjoint matrix such that $Y = A^2$ *for some* $A = A^H$ *and Y has eigenvalues on the nonpositive real half-axis. Assume further that*

($*$) *either Y has negative eigenvalues, or Y has no negative eigenvalues and the nonzero subspace* Ker Y *is not H-definite* (*i.e.*, $[x, x] = 0$ *for some nonzero* $x \in \operatorname{Ker} Y$).

Then there is a continuous family of matrices $Y(\alpha)$, $\alpha \in [0, 1]$, *with the following properties*:

(i) $Y(0) = Y$.

(ii) *Every* $Y(\alpha)$ *has the form* $Y(\alpha) = X(\alpha)^H X(\alpha)$ *for a suitable* $X(\alpha) \in F^{n\times n}$.

(iii) *Either all* $Y(\alpha)$ *are nonsingular, or all* $Y(\alpha)$ *are singular, and in the latter case the algebraic multiplicity of zero as an eigenvalue of* $Y(\alpha)$ *is constant, i.e., independent of* $\alpha \in [0, 1]$.

(iv) $Y(\alpha)$ *has no G-self-adjoint square root for any invertible Hermitian matrix G if* $\alpha > 0$.

If the hypothesis ($*$) *is not satisfied, then the conclusions of the theorem still hold with the conditions* (ii) *and* (iii) *replaced by the following ones*:

(ii′) *Every* $Y(\alpha)$ *is H-self-adjoint.*

(iii′) *The algebraic multiplicity of zero as an eigenvalue of* $Y(\alpha)$, $\alpha \neq 0$, *is one less than the algebraic multiplicity of zero as an eigenvalue of Y.*

We note that for the inverse statement of the theorem in the complex case, the existence of a matrix $Y(\alpha)$ with the properties described in (i)–(iv) in every neighborhood of Y follows from the results of [16], using the criteria for the representation in the form $X^H X$ given in Theorem 2.2 and for the existence of an H-self-adjoint square root given in Corollary 3.2. We prefer, however, to provide an explicit construction of the family $Y(\alpha)$. This construction will also be used in the next section.

The proof of Theorem 3.3 will show that in all cases the family $Y(\alpha)$ can be chosen in the form $Y(\alpha) = Y + \alpha B$, where B is a suitable H-self-adjoint matrix.

Proof. Let Y be an H-self-adjoint matrix such that $\sigma(Y) \cap (-\infty, 0] = \emptyset$. Then there exist three positively oriented simple Jordan contours Γ_+, Γ_{u} and Γ_{l} enclosing the eigenvalues of Y in $(0, +\infty)$, the open upper half-plane and the open lower half-plane, respectively. We also choose Γ_+ in the open right half-plane and symmetric with respect to the real line, Γ_{u} in the open upper half-plane, and Γ_{l} in the open lower half-plane such that, apart from its orientation, Γ_{u} becomes Γ_{l} upon reflection with respect to the real line. Suppose $\delta > 0$ is a constant such that any matrix Z satisfying $\|Y - Z\| < \delta$ has all of its eigenvalues in the interior regions of one of the above contours. Let $\mathrm{Sq}_+(z)$ denote the analytic function $\sqrt{z}$ satisfying $\mathrm{Sq}_+(1) = 1$ that has its branch cut along the halfaxis $(-\infty, 0]$. Given a G-self-adjoint matrix Z such that $\|Y - Z\| < \delta$, we now define

$$\begin{aligned} A = {} & \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} \mathrm{Sq}_+(z)(z - Y)^{-1} \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\mathrm{u}}} \mathrm{Sq}_+(z)(z - Y)^{-1} \, \mathrm{d}z \\ & + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\mathrm{l}}} \mathrm{Sq}_+(z)(z - Y)^{-1} \, \mathrm{d}z; \end{aligned}$$

$$\begin{aligned} B = {} & \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} \mathrm{Sq}_+(z)(z - Z)^{-1} \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\mathrm{u}}} \mathrm{Sq}_+(z)(z - Z)^{-1} \, \mathrm{d}z \\ & + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\mathrm{l}}} \mathrm{Sq}_+(z)(z - Z)^{-1} \, \mathrm{d}z. \end{aligned}$$

Then obviously $A^2 = Y$ and $B^2 = Z$. Moreover, the integrals over Γ_+ are H-self-adjoint and G-self-adjoint, respectively, and represent matrices having only positive eigenvalues, whereas the corresponding integrals over Γ_{u} and Γ_{l} are each other's H-adjoints (resp. G-adjoints) and represent matrices having their eigenvalues in the first and fourth quadrant, respectively. Consequently, A is H-self-adjoint and B is G-self-adjoint. Finally, (3.2) holds for sufficiently small $\delta > 0$.

For the converse, let Y have either negative or zero eigenvalues. Without loss of generality, we assume that (Y, H) is in the canonical form (1.4), (1.6) or (1.5), (1.6) with respect to some basis $\{e_{ij}\}_{i=1}^{\beta}{}_{j=1}^{m_i}$ in F^n.

By Theorem 3.1, the part of (Y,H) corresponding to each eigenvalue $-\beta^2$ with $\beta>0$ is $(Z,Q)=(\oplus_{i=1}^r(J_{k_i}(-\beta^2)\oplus J_{k_i}(-\beta^2)),\oplus_{i=1}^r(Q_{k_i}\oplus(-Q_{k_i})))$, where $k_1\leqslant\cdots\leqslant k_r$, with respect to some basis $\{e_{i,j}\}_{i=1}^{r}{}_{j=1}^{k_i}$. Choosing distinct $t_1,\ldots,t_r>0$, in view of the same Theorem 3.1, the perturbation $(Z(\alpha),Q)=(\oplus_{i=1}^r((1+t_i\alpha)J_{k_i}(-\beta^2)\oplus(1-t_i\alpha)J_{k_i}(-\beta^2)),\oplus_{i=1}^r(Q_{k_i}\oplus(-Q_{k_i})))$ produces a matrix that fails to have a G-self-adjoint square root for any G, for sufficiently small $\alpha>0$. (The numbers $t_1,\ldots,t_r$ are distinct to avoid producing a canonical form that on rearrangement of Jordan blocks satisfies the conditions of Theorem 3.1.) Observe that, by Theorem 2.2, $Z(\alpha)$ has the form X^QX for some X.

Now consider the case when $\sigma(Y)=\{0\}$. By Theorem 3.1, (Y,H) is built of the following components:

(a) $(J_m(0)\oplus J_m(0),Q_m\oplus(-Q_m))$ for $m\geqslant 2$.

(b) $(J_m(0)\oplus J_{m-1}(0),\varepsilon(Q_m\oplus Q_{m-1}))$ for $m\geqslant 2$ and $\varepsilon=\pm 1$.

(c) $(J_1(0)\oplus\cdots\oplus J_1(0),\operatorname{diag}(\varepsilon_1,\cdots,\varepsilon_p))$, where $\varepsilon_j=\pm 1$.

Assuming the condition $(*)$ holds true, a nilpotent H-self-adjoint perturbation $Y(\alpha)$ that yields the result of the converse statement of Theorem 3.3 will be constructed for each case separately.

Consider the case (a). Let $Y_0=J_m(0)\oplus J_m(0)$ and $H_0=Q_m\oplus(-Q_m)$ where $m\geqslant 2$. We assume in addition that m is the maximal partial multiplicity of Y. Define $Y_0(\alpha)=J_m(0)\oplus J_m(0)+T(\alpha)$, where $T(\alpha)$ is the square matrix of order $2m$ with α as its $(1,m+1)$ element, $-\alpha$ as its $(2m,m)$ element and zeros elsewhere, and take $\alpha\in\mathbb{R}$. Denote by $e_1,e_2,\ldots,e_{2m}$ the standard unit vectors in F^{2m}. The matrix $Y_0(\alpha)$ is nilpotent and H_0-self-adjoint with, for $\alpha\neq 0$ and $m>2$, the linearly independent maximal Jordan chains

$$e_m,e_{m-1}-\alpha e_{2m},e_{m-2}-\alpha e_{2m-1},\ldots,e_1-\alpha e_{m+2},-\alpha e_{m+1},-(\alpha)^2e_1;$$
$$-\alpha e_{m-1}+e_{2m-2},\ldots,-\alpha e_3+e_{m+2},-\alpha e_2+e_{m+1},$$

of length $m+2$ and $m-2$, respectively, where the eigenvector comes first. For $\alpha\neq 0$ and $m=2$ we have only the Jordan chain $\{-(\alpha)^2e_1,-\alpha e_3,e_1-\alpha e_4,e_2\}$. Let $Y(\alpha)$ be obtained from Y by replacing exactly one of the components $J_m(0)\oplus J_m(0)$ with $Y_0(\alpha)$. By Corollary 3.2, $Y(\alpha)$ does not have a G-self-adjoint square root for any G if $\alpha\neq 0$.

Consider the case (b). Assume that m is the maximal partial multiplicity of Y, and that there are no components of type (a) with the same value of m (if there are such components, then we are back to the above consideration of the case (a)). Let $Y_0=J_m(0)\oplus J_{m-1}(0)$ with $m\geqslant 2$, and let $H_0=\varepsilon(Q_m\oplus Q_{m-1})$. Define $Y_0(\alpha)=Y_0+\alpha\hat{T}$, where $\alpha\in\mathbb{R}$ and $\hat{T}$ has a one in the $(1,m+1)$ and $(2m-1,m)$ positions and zeros elsewhere. One checks that $Y_0(\alpha)$ is H_0-self-adjoint. The matrix $Y_0(\alpha)$ is nilpotent and H_0-self-adjoint with, for $\alpha\neq 0$ and $m>2$, the linearly independent maximal Jordan chains

$$\alpha^2 e_1, e_1 + \alpha e_{m+1}, e_2 + \alpha e_{m+2}, \ldots, e_{m-1} + \alpha e_{2m-1}, e_m;$$
$$e_{m+1} - \alpha e_2, e_{m+2} - \alpha e_3, \ldots, e_{2m-2} - \alpha e_{m-1},$$

of length $m+1$ and $m-2$, respectively, where the eigenvector comes first. For $\alpha \neq 0$ and $m = 2$ we have only the Jordan chain $\{\alpha^2 e_1, e_1 + \alpha e_3, e_2\}$. For $\alpha \neq 0$ the canonical form of the pair $(Y_0(\alpha), H_0)$ is easily seen to be $(J_{m+1}(0) \oplus J_{m-2}(0), \varepsilon(Q_{m+1} \oplus Q_{m-2}))$ if $m > 2$ and $(J_3(0), \varepsilon Q_3)$ if $m = 2$. Let q be the number of components of type (b) in (Y, H) with the index m equal to the maximal partial multiplicity of Y. If q is odd, then by applying the above perturbation for each one of these q components, we obtain a continuous family $Y(\alpha)$ such that $Y(0) = Y$ and for nonzero α the matrix $Y(\alpha)$ is nilpotent with an odd number of partial multiplicities equal to $m+1$ and no partial multiplicities equal to m. By Corollary 3.2, such $Y(\alpha)$ cannot have a G-self-adjoint square root for any G. If q is even, then we apply the following perturbation to exactly two of these q components:

$$(J_m(0) \oplus J_{m-1}(0) \oplus J_m(0) \oplus J_{m-1}(0), \varepsilon_1(Q_m \oplus Q_{m-1}) \oplus \varepsilon_2(Q_m \oplus Q_{m-1})).$$

Namely, we replace $Z = J_m(0) \oplus J_{m-1}(0) \oplus J_m(0) \oplus J_{m-1}(0)$ by $Z(\alpha) = Z + \alpha S$, where the $(4m-2) \times (4m-2)$ matrix S has 1 in the $(1, m+1)$, $(2m-1, m)$ and $(1, 2m)$ positions, $\varepsilon_1\varepsilon_2$ in the $(3m-1, m)$ position, and zeros elsewhere. It is not difficult to check that $Z(\alpha)$, $\alpha \neq 0$, is nilpotent, H-self-adjoint, and has partial multiplicities $m+2, m-1, m-1, m-2$. Indeed, denoting $\pm = \varepsilon_1\varepsilon_2$, $Z(\alpha)$ with $\alpha \neq 0$ and $m > 2$ has the independent maximal Jordan chains

$$\pm \alpha^2 e_1, \alpha^2 e_1 \pm \alpha e_{2m}, e_1 + \alpha e_{m+1} \pm \alpha e_{2m+1}, \ldots, e_{m-1} + \alpha e_{2m-1} \pm \alpha e_{3m-1}, e_m;$$
$$- e_{2m-1} + e_{3m-2}, \ldots, -e_{m+1} + e_{2m};$$
$$- \alpha e_{m-1} + e_{2m-2}, \ldots, -\alpha e_2 + e_{m+1};$$
$$e_{3m}, e_{3m+1}, \ldots, e_{4m-3}, e_{4m-2},$$

where the eigenvectors are given first and the lengths are $m+2$, $m-1$, $m-2$ and $m-1$, respectively. For $\alpha \neq 0$ and $m = 2$ we have the Jordan chain $\{e_2, e_1 + \alpha e_3 \pm \alpha e_5, \alpha^2 e_1 \pm \alpha e_4, \pm\alpha^2 e_1\}$ and the eigenvectors $-e_3 + e_4$ and e_6. As a result, we obtain a nilpotent H-self-adjoint family of matrices $Y(\alpha)$ having only one Jordan block of size $m+2$, and all other blocks of size less than $m+1$ (for nonzero α). By Corollary 3.2, $Y(\alpha)$ has no G-self-adjoint square roots for any G, if $\alpha \neq 0$.

For the rest of the proof we assume that (Y, H) does not contain components of type (a) or (b).

Consider the case where the pair (Y, H) is given by (c), and assume first that the hypothesis $(*)$ holds true. Then not all the signs ε_j are the same. In particular, $p \geqslant 2$. If $p = 2$, by applying a congruence to H, we assume H to have the form

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then

$$Y(\alpha) = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

will do. If $p \geqslant 3$, then select three signs among the ε_j that are not the same, say ε_j, $j = 1, 2, 3$. Applying a congruence to H, we may take H in the form

$$H = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \operatorname{diag}(\varepsilon_4, \ldots, \varepsilon_p).$$

Then $Y(\alpha) = \alpha W$, where W has 1 in the $1, 2$ and $2, 3$ positions and zeros everywhere else will do. Finally, consider the case (c) with the hypothesis (*) not satisfied. Then all the signs ε_j are the same, and the continuous family $Y(\alpha) = J_1(-\alpha) \oplus J_1(0) \oplus J_1(0) \oplus \cdots \oplus J_1(0)$ satisfies the conditions (i), (ii′), (iii′) and (iv). □

We say that an H-self-adjoint square root A of an H-self-adjoint matrix Y is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that a G-self-adjoint matrix Z has a G-self-adjoint square root B satisfying $\|B - A\| < \varepsilon$, as soon as $\|Z - Y\| < \delta$. Theorem 3.3 shows in particular that there exists a stable H-self-adjoint square root of Y if and only if $\sigma(Y) \cap (-\infty, 0] = \emptyset$. Moreover, in this case an H-self-adjoint square root A of Y is stable if and only if A and $-A$ no common eigenvalues, and then A is a real analytic function of Y. This can be proved without difficulty using an integral representation of A similar to the integral formula used in the first part of the proof of Theorem 3.3.

4. Stability of H-polar decompositions

In this section we derive necessary and sufficient conditions on an $n \times n$ matrix X over F to have an H-polar decomposition $X = UA$ with an H-unitary factor U and an H-self-adjoint factor A that depend continuously on X. We consider here the case when $X^H X$ has negative eigenvalues or is singular with some Jordan blocks corresponding to the zero eigenvalue have size larger than one. The next section is devoted to the consideration of the remaining cases.

To find an H-polar decomposition of a given $n \times n$ matrix X over F one must construct an H-self-adjoint matrix A satisfying (1.2). The H-unitary matrices U appearing in the H-polar decomposition $X = UA$ then are the H-unitary extensions of the H-isometry $V : \operatorname{Im} A \to \operatorname{Im} X$ defined by $V(Au) = Xu$. Our strategy in proving the main stability result of this section is to construct an H-self-adjoint matrix A satisfying (1.2) that depends continuously on X, if possible.

Theorem 4.1. *Let X be an $n \times n$ matrix over F such that $\sigma(X^H X) \cap (-\infty, 0] = \emptyset$. Then there exist an H-self-adjoint matrix A satisfying* (1.2), *an H-unitary matrix U satisfying* (1.1), *and constants $\delta, M > 0$, depending on X, A, H, and U only, such that for any pair of $n \times n$ matrices (Y, G) over F with G nonsingular self-adjoint and $\|X - Y\| + \|G - H\| < \delta$ there exists a G-polar decomposition $Y = VB$ of X satisfying*

$$\|A - B\| + \|U - V\| \leqslant M[\|X - Y\| + \|H - G\|]. \tag{4.1}$$

Moreover, such an A can be chosen with the additional property that $\sigma(A) \cap (-\infty, 0] = \emptyset$.

Conversely, let X be an $n \times n$ matrix over F having an H-polar decomposition and such that one of the following three conditions are satisfied:

(α) *$X^H X$ has negative eigenvalues*;

(β) $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$ *and* $\operatorname{Ker} X^H X \neq \operatorname{Ker}(X^H X)^n$;

(γ) $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$ *and* $\operatorname{Ker}(X^H X)^n = \operatorname{Ker} X^H X \neq \operatorname{Ker} X$.

Then in every neighborhood of X there is an $n \times n$ matrix Y over F such that Y does not have an H-polar decomposition. Moreover, Y can be chosen so that $Y^H Y$ does not have a G-self-adjoint square root for any invertible self-adjoint matrix G.

The case which is left out of Theorem 4.1, namely, when $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$ and the subspace $\operatorname{Ker} X^H X = \operatorname{Ker}(X^H X)^n = \operatorname{Ker} X$, will be considered in the next section.

Proof. First let X have an H-polar decomposition, and assume that $X^H X$ does not have zero or negative eigenvalues. Then by Theorem 3.3 there exists an H-self-adjoint matrix A satisfying $A^2 = X^H X$ and constants $\delta', M' > 0$ such that for any G-self-adjoint matrix Z with $\|X^H X - Z\| + \|G - H\| < \delta'$ there exists a G-self-adjoint matrix B satisfying $B^2 = Z$ such that

$$\|A - B\| \leqslant M' \|X^H X - Z\|. \tag{4.2}$$

Note the identity

$$\begin{aligned} H^{-1} X^* H X - G^{-1} Y^* G Y = {} & H^{-1}[X^* H(X - Y) \\ & + X^*(H - G)(Y - X) + X^*(H - G)X + (X^* - Y^*)(G - H)(Y - X) \\ & + (X^* - Y^*)(G - H)X + (X^* - Y^*)H(Y - X) \\ & + (X^* - Y^*)HX] + (H^{-1} - G^{-1}) \\ & \times [(Y^* - X^*)(G - H)(Y - X) + (Y^* - X^*)(G - H)X \\ & + (Y^* - X^*)HY + X^* G Y], \end{aligned}$$

which yields the estimate

$$\|X^H X - Y^G Y\| \tag{4.3}$$

$$\leqslant 4\left[\|H\| + \|H^{-1}\|^{-1}\right]\left[\max(1, \|H\|, \|H^{-1}\|, \|X\|)\right]^4$$

$$\times [\|X - Y\| + \|H - G\|] \tag{4.4}$$

whenever $\|H - G\| \leqslant \frac{1}{2}\|H^{-1}\|^{-1}$. Now apply the inequality (4.2) with $Z = Y^G Y$. Taking $\delta > 0$ sufficiently small, in view of (4.3) we can guarantee the inequality $\|X^H X - Z\| + \|G - H\| < \delta'$, and therefore by (4.2), $\|A - B\| \leqslant M''[\|X - Y\| + \|H - G\|]$ for some constant M'' that depends on X and H only. Now (4.1) follows using the formulae $U = XA^{-1}$, $V = YB^{-1}$.

For the converse, assume first that at least one of (α) and (β) is satisfied. Thus, let $X^H X$ have either negative or zero eigenvalues; if $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$, it will be assumed that the subspace $\operatorname{Ker} X^H X$ is strictly smaller than $\operatorname{Ker}(X^H X)^n$. Let $(X^H X, H)$ be in the canonical form (1.4) and (1.6) or (1.5) and (1.6) with respect to some basis $\{e_{ij}\}_{i=1}^{\beta}{}_{j=1}^{m_i}$. Our strategy will be to consider one block at the time, say the block indexed by i_0, and to only perturb the action of X on $\{e_{i_0 j}\}_{j=1}^{m_{i_0}}$ while not changing its action on the vectors e_{ij}, $i \neq i_0$. We will also need to consider several such blocks simultaneously, and to perturb the action of X on these blocks only. The following lemma guarantees that such perturbations do not affect the action of $X^H X$ on all other blocks:

Lemma 4.2. *Assume that $(X^H X, H)$ is in the canonical form* (1.4) *and* (1.6) *or* (1.5) *and* (1.6) *with respect to some basis $\{e_{ij}\}_{i=1}^{\beta}{}_{j=1}^{m_i}$. Select an index i_0, and let a matrix $\tilde{X}$ be defined in such a way that for every j, the vector $\tilde{X}e_{i_0 j}$ is a linear combination of $Xe_{i_0 1}, \ldots, Xe_{i_0, m_{i_0}}$, whereas $\tilde{X}e_{ik} = Xe_{ik}$ if $i \neq i_0$. Then*

$$\tilde{X}^H \tilde{X} e_{ik} = X^H X e_{ik} \tag{4.5}$$

for all $i \neq i_0$ and all $k = 1, \ldots, m_i$.

Proof. Let $i \neq i_0$. Then if $i_1 \neq i_0$ we obtain:

$$[\tilde{X}^H \tilde{X} e_{ij}, e_{i_1 k}] = [\tilde{X} e_{ij}, \tilde{X} e_{i_1 k}] = [Xe_{ij}, Xe_{i_1 k}] = [X^H X e_{ij}, e_{i_1 k}]. \tag{4.6}$$

For the selected index i_0 and for certain scalars $\alpha_{k1}, \ldots, \alpha_{km_{i_0}} \in F$ we have

$$[\tilde{X}^H \tilde{X} e_{ij}, e_{i_0 k}] = [\tilde{X} e_{ij}, \tilde{X} e_{i_0 k}] = \left[Xe_{ij}, \sum_{p=1}^{m_{i_0}} \alpha_{kp} X e_{i_0 p}\right]$$

$$= \left[X^H X e_{ij}, \sum_{p=1}^{m_{i_0}} \alpha_{kp} e_{i_0 p}\right] = 0; \tag{4.7}$$

here the last equality follows from the canonical form. But also

$$[X^H X e_{ij}, e_{i_0 k}] = 0, \tag{4.8}$$

again because of the canonical form. Now clearly (4.5) follows from (4.6), (4.7), and (4.8). □

Proof of Theorem 4.1 (*continued*). Let X have an H-polar decomposition and suppose first that $X^H X$ has negative eigenvalues. Then the part of $(X^H X, H)$ (which is already assumed to be in the canonical form with respect to the basis $\{e_{ij}\}_{j=1}^{m_i}$, $i = 1, \ldots, p$) corresponding to each eigenvalue $-\beta^2$ with $\beta > 0$ is an orthogonal sum of blocks of the type $(J_k(-\beta^2) \oplus J_k(-\beta^2), Q_k \oplus (-Q_k))$. Select an index i_0 such that the part of $(X^H X, H)$ with respect to the basis $\{e_{i_0 j}\}_{j=1}^{m_{i_0}}$ in the corresponding block is given by $(J_{m_{i_0}}(-\beta^2), Q_{m_{i_0}})$, and define $\tilde{X}$ by $\tilde{X} e_{i_0 j} = \sqrt{1+\varepsilon} X e_{i_0 j}$, where $\varepsilon > 0$ is small enough, and by $\tilde{X} e_{ij} = X e_{ij}$ for $i \neq i_0$. In view of Lemma 4.2, the canonical form of $(\tilde{X}^H \tilde{X}, H)$ is the same as the canonical form of $(X^H X, H)$, except that the block $(J_{m_{i_0}}(-\beta^2), Q_{m_{i_0}})$ is replaced by $((1+\varepsilon) J_{m_{i_0}}(-\beta^2), Q_{m_{i_0}})$. By Theorem 3.1, the matrix $\tilde{X}^H \tilde{X}$ does not have an H-self-adjoint square root (for ε sufficiently close to zero). This completes case (α).

Next, assume we are in case (β), that is, X is singular, with $\operatorname{Ker}(X^H X)$ not an H-definite subspace, $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$, and X has an H-polar decomposition. Then $X^H X$ has an H-self-adjoint square root A satisfying (1.2) and the part of the canonical form of $(X^H X, H)$ pertaining to the zero eigenvalue is described by Theorem 3.1 (ii). We will assume $(X^H X, H)$ to be in the canonical form, and construct a continuous perturbation $X(\alpha)$ of X such that $X^H(\alpha) X(\alpha)$ does not have a G-self-adjoint square root for any invertible Hermitian G. Since we have excluded the case when the Jordan blocks of $X^H X$ corresponding to the zero eigenvalue have all sizes equal to one, in view of Theorem 3.1 the canonical form of $(X^H X, H)$ must contain at least one of the following components:

(a) $(J_m(0) \oplus J_{m-1}(0), \varepsilon(Q_m \oplus Q_{m-1}))$ for $m \geqslant 3$ and $\varepsilon = \pm 1$.
(b) $(J_m(0) \oplus J_m(0), Q_m \oplus (-Q_m))$ for $m \geqslant 3$.
(c) $(J_2(0) \oplus J_1(0), \varepsilon(Q_2 \oplus Q_1))$ and $\varepsilon = \pm 1$.
(d) $(J_2(0) \oplus J_2(0), Q_2 \oplus (-Q_2))$.

For notational simplicity, it will be assumed that one of the components (a)–(d), as appropriate in each case, appears first in the canonical form (1.4) and (1.6) (or (1.5) and (1.6)), and the corresponding basis vectors will be denoted $e_1, e_2, \ldots$. For example, if the pair $(X^H X, H)$ contains the component (a), then we denote by $e_1, \ldots, e_{2m-1}$ the first $2m-1$ basis vectors, and with respect to the linearly independent vectors $e_1, \cdots, e_{2m-1}$ the matrices $X^H X$ and H have the form $J_m(0) \oplus J_{m-1}(0)$ and $\varepsilon(Q_m \oplus Q_{m-1})$, respectively. The remaining basis vectors (in case $(X^H X, H)$ contains the component (a)) will be denoted

$e_{2m}, \ldots, e_n$, and analogously in the cases when $(X^H X, H)$ contains one of the components (b)–(d).

Consider the case (a). We assume that $m \geqslant 3$ is the maximal partial multiplicity of $X^H X$ associated with the zero eigenvalue, and that $(X^H X, H)$ does not have components of type (b) with the same m (otherwise, we will consider $(X^H X, H)$ as satisfying the case (b); see below). We further assume that the number q of components (a) with the same value of m is odd. Thus, the part of $(X^H X, H)$ corresponding to the first $(2m-1)q$ basis vectors has the form

$$(Z_0, H_0) = \left(\overset{q}{\underset{j=1}{\oplus}} (J_m(0) \oplus J_{m-1}(0)),\ \overset{q}{\underset{j=1}{\oplus}} \varepsilon_j (Q_m \oplus Q_{m-1}) \right).$$

Define $Z_0(\alpha) = Z_0 + \oplus_{j=1}^q (\alpha T)$, where $\alpha \in \mathbb{R}$ and T is the $(2m-1) \times (2m-1)$ matrix having a one in the $(1, m+1)$ and $(2m-1, m)$ positions and zeros elsewhere. Let $Z(\alpha)$ be the $n \times n$ matrix whose part with respect to $\{e_1, \ldots, e_{(2m-1)q}\}$ is $Z_0(\alpha)$ and which coincides with $X^H X$ with respect to $\{e_{(2m-1)q+1}, \ldots, e_n\}$. In other words, $Z(\alpha)$ is equal to the perturbation $Y(\alpha)$ described in the case (b) of the proof of Theorem 3.3. Since $Y(\alpha)$ (for $\alpha \neq 0$) does not admit a G-self-adjoint square root for any G, as we have seen in the proof of Theorem 3.3, to complete the consideration of the case (a) with odd q, we only have to find $X(\alpha)$ depending continuously on α such that $X(\alpha)^H X(\alpha) = Z(\alpha)$ and $X(0) = X$. Such an $X(\alpha)$ is given as follows: For each block of vectors $f_{kj} = e_{(2m-1)k+j}$, $j = 1, \ldots, 2m-1$, where $k = 0, \ldots, q-1$ is fixed, define $X(\alpha) f_{kj} = X f_{kj}$ for $j \neq m+1$, and fix $X(\alpha) f_{k,m+1}$ by the requirement that $X(\alpha)(-\alpha f_{k2} + f_{k,m+1}) = X f_{k,m+1}$, i.e., $X(\alpha) f_{k,m+1} = X f_{k,m+1} + \alpha X f_{k2}$. Clearly $X(0) = X$. It remains to check that $X(\alpha)^H X(\alpha) = Z(\alpha)$.

Fix $k \in \{1, \ldots, q-1\}$. We denote by δ_{pq} the Kronecker symbol: $\delta_{pq} = 0$ if $p \neq q$, $\delta_{pq} = 1$ if $p = q$. Note that

$$H[Z(0) - Z(\alpha)] f_{k,j} = -\varepsilon\alpha \big(\delta_{j,m+1} f_{k,m} + \delta_{j,m} f_{k,m+1} \big)$$

and $X(\alpha) f_{k,j} = X \big(f_{k,j} + \alpha \delta_{j,m+1} f_{k2} \big)$. As a result,

$$\begin{aligned} \langle H X(\alpha) f_{k,j}, X(\alpha) f_{k,r} \rangle &= \langle H Z(0) \big(f_{k,j} + \alpha \delta_{j,m+1} f_{k2} \big), (f_{k,r} + \alpha \delta_{r,m+1} f_{k2}) \rangle \\ &= \langle H Z(0) f_{k,j}, f_{k,r} \rangle + \varepsilon\alpha \delta_{j,m+1} \langle f_{k,m}, f_{k,r} \rangle + \varepsilon\alpha \delta_{r,m+1} \langle f_{k,j}, f_{k,m} \rangle \\ &\quad + \varepsilon\alpha^2 \delta_{j,m+1} \delta_{r,m+1} \langle f_{k,m}, f_{k2} \rangle, \end{aligned}$$

where the last term vanishes because $m > 2$. Consequently,

$$\begin{aligned} &\langle H X(\alpha) f_{k,j}, X(\alpha) f_{k,r} \rangle - \langle H Z(\alpha) f_{k,j}, f_{k,r} \rangle \\ &\quad = -\varepsilon\alpha \big\{ \delta_{j,m+1} \langle f_{k,m}, f_{k,r} \rangle + \delta_{j,m} \langle f_{k,m+1}, f_{k,r} \rangle \big\} \\ &\qquad + \varepsilon\alpha \big\{ \delta_{j,m+1} \langle f_{k,m}, f_{k,r} \rangle + \delta_{r,m+1} \langle f_{k,j}, f_{k,m} \rangle \big\} = 0, \end{aligned}$$

which implies $X(\alpha)^H X(\alpha) = Z(\alpha)$, as claimed.

Next, we consider the case (a), still assuming that m is the maximal partial multiplicity of X^HX associated with the zero eigenvalue, but now the number q of components of type (a) with the same m is even. For simplicity of notation, let $q=2$. Letting $X^HX = J_m(0) \oplus J_{m-1}(0) \oplus J_m(0) \oplus J_{m-1}(0)$ and $H = \varepsilon_1(Q_m \oplus Q_{m-1}) \oplus \varepsilon_2(Q_m \oplus Q_{m-1})$ with respect to the basis vectors $e_1, \ldots, e_{4m-2}$, we define $X(\alpha)e_j = X(e_j + \alpha[\delta_{j,m+1} + \delta_{j,2m}]e_2)$, where $m > 2$. Now note that

$$H[Z(0) - Z(\alpha)]e_j = -\varepsilon_1 \alpha \delta_{j,m}(e_{m+1} + e_{2m}) - \varepsilon_1 \alpha \big(\delta_{j,m+1} + \delta_{j,2m}\big) e_m.$$

As a result,

$$\begin{aligned}
&\langle HX(\alpha)e_j, X(\alpha)e_r\rangle \\
&\quad = \langle HZ(0)\big(e_j + \alpha\big[\delta_{j,m+1} + \delta_{j,2m}\big]e_2\big), e_r + \alpha[\delta_{r,m+1} + \delta_{r,2m}]e_2\rangle \\
&\quad = \langle HZ(0)e_j, e_r\rangle + \varepsilon_1\alpha\big[\delta_{j,m+1} + \delta_{j,2m}\big]\langle HZ(0)e_2, e_r\rangle \\
&\qquad + \varepsilon_1\alpha[\delta_{r,m+1} + \delta_{r,2m}]\langle e_j, HZ(0)e_2\rangle \\
&\qquad + \alpha^2\big[\delta_{j,m+1} + \delta_{j,2m}\big][\delta_{r,m+1} + \delta_{r,2m}]\langle HZ(0)e_2, e_2\rangle \\
&\quad = \langle HZ(0)e_j, e_r\rangle + \varepsilon_1\alpha\big[\delta_{j,m+1} + \delta_{j,2m}\big]\langle e_m, e_r\rangle + \varepsilon_1\alpha[\delta_{r,m+1} + \delta_{r,2m}]\langle e_j, e_m\rangle,
\end{aligned}$$

where the term proportional to α^2 vanishes because $HZ(0)e_2 = e_m$ and $m > 2$. Consequently,

$$\begin{aligned}
&\langle HX(\alpha)e_j, X(\alpha)e_r\rangle - \langle HZ(\alpha)e_j, e_r\rangle \\
&\quad = -\varepsilon_1\alpha\big\{\delta_{j,m}[\langle e_{m+1}, e_r\rangle + \langle e_{2m}, e_r\rangle] + \big[\delta_{j,m+1} + \delta_{j,2m}\big]\langle e_m, e_r\rangle\big\} \\
&\qquad + \varepsilon_1\alpha\big\{\big[\delta_{j,m+1} + \delta_{j,2m}\big]\langle e_m, e_r\rangle + [\delta_{r,m+1} + \delta_{r,2m}]\langle e_j, e_m\rangle\big\} = 0,
\end{aligned}$$

which implies $X(\alpha)^H X(\alpha) = Z(\alpha)$, as claimed.

Consider the case (b). Thus, we may assume that the canonical form of the pair (X^HX, H) contains a block of the form $(J_m(0) \oplus J_m(0), Q_m \oplus (-Q_m))$ where $m \geq 3$, with respect to the basis vectors $e_1, \ldots, e_{2m}$. Assume in addition that m is the largest partial multiplicity of X^HX associated with the zero eigenvalue. Define $X(\alpha)e_j = Xe_j$ for $j \neq m+1$, $X(\alpha)e_{m+1} = Xe_{m+1} + \alpha Xe_2$, and $X(\alpha)e_p = Xe_p$ for $p > 2m$, where $\alpha \in \mathbb{R}$. Then $X(\alpha) \to X$ as $\alpha \to 0^+$. Moreover,

$$\begin{aligned}
&\langle HX(\alpha)^H X(\alpha)e_i, e_j\rangle = \langle HX(\alpha)e_i, X(\alpha)e_j\rangle \\
&\quad = \begin{cases}
\langle HX^HXe_i, e_j\rangle = \langle He_{i-1}, e_j\rangle = \delta_{i+j,m+2} - \delta_{i+j,3m+2}, & m+1 \notin \{i,j\} \\
\langle HX^HX(e_{m+1} + \alpha e_2), e_j\rangle = \alpha\langle He_1, e_j\rangle = \alpha\,\delta_{j,m}, & i = m+1,\ j \neq m+1 \\
\langle He_i, X^HX(e_{m+1} + \alpha e_2)\rangle = \alpha\langle He_i, e_1\rangle = \alpha\,\delta_{i,m}, & i \neq m+1,\ j = m+1 \\
\langle HX^HX(e_{m+1} + \alpha e_2), e_{m+1} + \alpha e_2\rangle = \alpha^2\langle He_1, e_2\rangle = 0, & i = j = m+1.
\end{cases}
\end{aligned}$$

It follows that the part of $X(\alpha)^H X(\alpha)$ corresponding to $e_1, \ldots, e_{2m}$ has the form $J_m(0) \oplus J_m(0) + \alpha T$, where T is the square matrix of order $2m$ with 1 as its $(1, m+1)$ entry, -1 as its $(2m, m)$ entry and zeros elsewhere. By Lemma 4.2, the part of $X(\alpha)^H X(\alpha)$ corresponding to $e_{2m+1}, \dots, e_n$ coincides with that of $X^H X$. In other words, $X(\alpha)^H X(\alpha) = Y(\alpha)$ where $Y(\alpha)$ is taken from the proof of the case (a) in Theorem 3.3. Since $Y(\alpha)$ has no H-self-adjoint square roots (if $\alpha \neq 0$), the same is true for $X(\alpha)^H X(\alpha)$.

For the rest of the proof, it will be assumed that the nilpotent part of $X^H X$ has no Jordan blocks of size larger than two.

Consider the case (c) *where* $\varepsilon = 1$. Without loss of generality we can again assume that the corresponding sign is $+1$. Now the pair $(X^H X, H)$ contains a pair $(J_2(0) \oplus J_1(0), Q_2 \oplus Q_1)$ with respect to the basis vectors e_1, e_2, e_3. (These vectors are part of the standard basis $e_1, \dots, e_n$ in F^n). Also, since X has an H-polar decomposition, according to (1.2) and Theorem 3.1 (e), we have

$$\operatorname{Ker} X \cap \operatorname{span}\{e_1, e_2, e_3\} = \operatorname{span}\{e_1\}.$$

Now note that $\langle HXe_j, Xe_r\rangle = \delta_{j,2}\delta_{j,r}$ for $1 \leqslant j, r \leqslant 3$, so the subspace $\operatorname{span}\{Xe_2\}$ is H-positive. Hence (see, e.g., Proposition 1.1 in [13]) the H-orthogonal complement $\mathscr{N}$ of $\operatorname{span}\{Xe_2\}$ is H-nondegenerate, H-indefinite, and contains the nonzero H-neutral vector Xe_3. Therefore $\mathscr{N}$ contains a vector v satisfying the conditions

$$\langle Hv, Xe_3\rangle = -1, \qquad \langle Hv, Xe_j\rangle = 0 \quad \text{for } j = 4, \dots, n,$$

so that $\langle HXe_i, v\rangle = \langle Hv, Xe_i\rangle = -\delta_{i,3}$ $(1 \leqslant i \leqslant n)$. Define $X(\alpha)$ by $X(\alpha)e_j = Xe_j + \alpha\delta_{j,3}v$. Then $X(\alpha) \to X$ as $\alpha \to 0^+$. Moreover, for $1 \leqslant j, r \leqslant n$ we have

$$\begin{aligned}\langle HX(\alpha)^H X(\alpha)e_j, e_r\rangle &= \langle H\left[Xe_j + \alpha\delta_{j,3}v\right], Xe_r + \alpha\delta_{r,3}v\rangle \\ &= \langle HX^H Xe_j, e_r\rangle + \alpha\delta_{j,3}\langle Hv, Xe_r\rangle + \alpha\delta_{r,3}\langle HXe_j, v\rangle + \alpha^2\delta_{j,3}\delta_{r,3}\langle Hv, v\rangle \\ &= \langle HX^H Xe_j, e_r\rangle - \delta_{j,3}\delta_{r,3}(2\alpha - \alpha^2\langle Hv, v\rangle).\end{aligned}$$

It follows that $X(\alpha)^H X(\alpha)e_j = X^H Xe_j$ for $j \neq 3$, and

$$X(\alpha)^H X(\alpha)e_3 = X^H Xe_3 + (-2\alpha + \alpha^2\langle Hv, v\rangle)e_3.$$

Therefore, $X(\alpha)^H X(\alpha)$ has the simple eigenvalue $-2\alpha + \alpha^2\langle Hv, v\rangle$, and hence by Theorem 3.1, for α sufficiently close to zero, $X(\alpha)^H X(\alpha)$ does not have a G-self-adjoint square root for any G. When the canonical form of $(X^H X, H)$ contains several components of the type of case (c), it suffices to apply the above perturbation to exactly one of these components and to no other component of the canonical form, in order to create a matrix $X(\alpha)$ such that $X(\alpha)^H X(\alpha)$ does not have a G-self-adjoint square root for any $\alpha > 0$ sufficiently close to zero.

Consider the case (d). With respect to the basis vectors e_1, e_2, e_3, e_4, the pair $(X^H X, H)$ is given by $(J_2(0) \oplus J_2(0), Q_2 \oplus (-Q_2))$, while X is assumed to have an H-polar decomposition. Then there exists an H-self-adjoint matrix A such that $X^H X = A^2$ and

$$\operatorname{Ker} X \cap \operatorname{span}\{e_1, e_2, e_3, e_4\}$$
$$= \operatorname{Ker} A \cap \operatorname{span}\{e_1, e_2, e_3, e_4\} = \operatorname{span}\{e_1 + e_3\}.$$

Now note that

$$\langle HXe_j, Xe_r\rangle = [\delta_{j,2} - \delta_{j,4}]\delta_{j,r}, \quad 1 \leqslant j, r \leqslant 4.$$

Therefore, the subspace $\operatorname{span}\{Xe_2, Xe_4\}$ is H-nondegenerate and H-indefinite. Hence its H-orthogonal complement $\mathscr{N}$ contains a vector v satisfying

$$\langle Hv, Xe_1\rangle = \langle HXe_1, v\rangle = -\langle Hv, Xe_3\rangle = -\langle HXe_3, v\rangle = 1,$$

$$\langle Hv, Xe_j\rangle = 0 \quad \text{for } j = 5, \ldots, n,$$

where we note that $Xe_1 = -Xe_3$. Hence $\langle Hv, Xe_i\rangle = \langle HXe_i, v\rangle = \delta_{i,1} - \delta_{i,3}$. For $\alpha \geqslant 0$ define $X(\alpha)$ by

$$X(\alpha)e_j = Xe_j + \alpha(\delta_{j,1} - \delta_{j,3})v, \quad 1 \leqslant j \leqslant n.$$

We then easily compute for $1 \leqslant j, r \leqslant n$:

$$\begin{aligned}\langle HX(\alpha)^H X(\alpha)e_j, e_r\rangle &= \langle H\left[Xe_j + \left(\delta_{j,1} - \delta_{j,3}\right)\alpha v\right], Xe_r + (\delta_{r,1} - \delta_{r,3})\alpha v\rangle \\ &= \langle HX^H Xe_j, e_r\rangle + \left(\delta_{j,1} - \delta_{j,3}\right)\alpha\langle Hv, Xe_r\rangle + (\delta_{r,1} - \delta_{r,3})\alpha\langle HXe_j, v\rangle \\ &\quad + \alpha^2\left(\delta_{j,1} - \delta_{j,3}\right)(\delta_{r,1} - \delta_{r,3})\langle Hv, v\rangle \\ &= \langle HX^H Xe_j e_r\rangle + (\delta_{j,1} - \delta_{j,3})(\delta_{r,1} - \delta_{r,3})(2\alpha + \alpha^2\langle Hv, v\rangle).\end{aligned}$$

As a result, the part of $X(\alpha)^H X(\alpha)$ with respect to the basis vectors e_1, e_2, e_3 and e_4 has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ q & 0 & -q & 0 \\ 0 & 0 & 0 & 1 \\ q & 0 & -q & 0 \end{pmatrix}, \tag{4.9}$$

where $q = 2\alpha + \alpha^2\langle Hv, v\rangle$, whereas the remaining part of $X(\alpha)^H X(\alpha)$ is exactly the same as the corresponding part of $X^H X$. The matrix (4.9) is a singular matrix having $\{q(e_1 + e_3), q(e_2 + e_4), e_1, e_3\}$ as its only Jordan chain (provided $q \neq 0$). By Theorem 3.1 it follows that $X(\alpha)^H X(\alpha)$ has no G-self-adjoint square root for any invertible Hermitian G if $\alpha > 0$ is sufficiently close to zero.

This concludes the proof of Theorem 4.1, under the hypothesis that (β) is satisfied.

Finally, assume that (γ) holds true. We use a different approach here. Let $X = UA$ be an H-polar decomposition of X. Note that $\operatorname{Ker} X = \operatorname{Ker} A$ and $X^H X = A^2$. Because of the hypothesis (γ), the canonical form of $(X^H X, H)$ contains the blocks

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

and we may assume $\operatorname{Ker} X = \operatorname{span}\{e_1 + e_2\}$. Changing the basis, we may in fact assume that the canonical form of the pair (A, H) contains the blocks

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right). \tag{4.10}$$

Let $\tilde{A}$ be the matrix obtained from A by replacing the block

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of (4.10) with

$$\begin{pmatrix} \varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix},$$

where ε is a nonzero real number close to zero, and let $Y = U\tilde{A}$. Then Y is close to X. A simple computation shows that $Y^H Y$ has exactly one Jordan block of size two corresponding to the negative eigenvalue $-\varepsilon^2$. In view of Theorem 3.1, $Y^H Y$ does not have a G-self-adjoint square root for any G. □

5. Stability of H-polar decompositions: the remaining cases

In this section we continue to study stability of H-polar decompositions, and in particular, we take care of the cases left out of Theorem 4.1. In contrast with Theorem 4.1, we will produce perturbations of X that admit H-polar decompositions. Nevertheless, in many cases there are no stable H-polar decompositions.

An H-polar decomposition $X = UA$ is called *H-stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every matrix $Y \in F^{n\times n}$ admits an H-polar decomposition $Y = VB$ with $\|U - V\| + \|A - B\| < \varepsilon$, as soon as $\|Y - X\| < \delta$.

Theorem 5.1. *Assume that $X \in F^{n\times n}$ admits an H-polar decomposition, and assume furthermore that* $\dim \operatorname{Ker} X \geqslant 1$ *in the complex case or* $\dim \operatorname{Ker} X \geqslant 2$ *in the real case. Then no H-polar decomposition of X is H-stable.*

Proof. Let $X = UA$ be an H-polar decomposition of X. First of all, note that the set of all possible H-unitary U's in this H-polar decomposition (with a fixed A) is infinite. Indeed, such U is obtained as a Witt extension of an H-isometry $\hat{U} : \operatorname{Im} A \to \operatorname{Im} X$, and the set of all such Witt extensions is a continuum, under the hypotheses of the theorem (see Theorems 2.5 and 2.6 of [7]).

We now fix an invertible H-self-adjoint matrix D such that D^2 has n distinct eigenvalues. Then for $\alpha \in \mathbb{R} \setminus \{0\}$ close enough to zero, the matrix $A + \alpha D$ is H-self-adjoint and invertible, and $(A + \alpha D)^2$ has n distinct eigenvalues (the latter fact follows from the analytic perturbation theory of matrices, see, e.g., Chapter 19 in [14]). Let $X(\alpha) = U(A + \alpha D)$. Because $(A + \alpha D)^2$ has n distinct eigenvalues (for real $\alpha \neq 0$ close to zero), there exist only finitely many H-polar decompositions

$$X(\alpha) = U_j(\alpha)A(\alpha), \qquad j = 1, \ldots, q \tag{5.1}$$

of $X(\alpha)$. Analytic perturbation theory (see Theorem 19.2.4 of [14] or the more general results of Section IX.3 of [2]) guarantees that each one of $U_1(\alpha), \ldots, U_q(\alpha)$ can be expressed as a series $\sum_{k=-p}^{\infty} M_k \alpha^{k/m}$ with suitable matrix coefficients M_k, where p and m are positive integers. In particular, there are only finitely many matrices $U_1, \ldots, U_\ell$ that appear as limits $\lim_{\alpha_k \downarrow 0} U_{j_k}(\alpha_k)$ for any sequence $\alpha_k \downarrow 0$ and any choice of the factor $U_{j_k}(\alpha_k)$ in the H-polar decomposition (5.1).

Now select another H-polar decomposition $X = \tilde{U}A$, and let $\tilde{X}(\alpha) = \tilde{U}(A + \alpha D)$. Arguing as in the preceding paragraph, we obtain that only the matrices $\tilde{U}U^{-1}U_1, \ldots, \tilde{U}U^{-1}U_\ell$ appear as limit points of the H-unitary factors in the H-polar decompositions of $\tilde{X}(\alpha)$ (as $\alpha \downarrow 0$). Therefore, if an H-polar decomposition $X = VA$ were stable, we would have to have $V = \tilde{U}UU_g$ for every $\tilde{U}$ that appears in an H-polar decomposition $X = \tilde{U}A$ and for some $g \in \{1, \ldots, \ell\}$ (which may depend on $\tilde{U}$). Equivalently, $\tilde{U} = VU_g^{-1}U^{-1}$. But this is impossible, since the set of all admissible $\tilde{U}$'s is infinite. □

The results of the present and the preceding sections show that no H-polar decomposition $X = UA$ is H-stable as soon as $X^H X$ has negative or zero eigenvalues, with the possible exception of the situation when $F = \mathbb{R}$, $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$, the dimension of the kernel of X is one, and $\operatorname{Ker} X^H X = \operatorname{Ker} (X^H X)^n = \operatorname{Ker} X$. This exceptional situation requires special consideration. By analogy with the Hilbert space (see [15]) we expect here H-stability of H-polar decompositions. Indeed, we will prove below that the exceptional situation admits H-stable H-polar decompositions. The proof will use the same ideas as the proof of Theorem 3.1 in [15].

Theorem 5.2. *Let $F = \mathbb{R}$. Assume that X admits H-polar decomposition, $\sigma(X^H X) \cap (-\infty, 0] = \{0\}$, and $\operatorname{Ker} X^H X = \operatorname{Ker} (X^H X)^n = \operatorname{Ker} X$ is one-dimensional. Then there exists an H-stable H-polar decomposition $X = UA$.*

Proof. The pair $(X^H X, H)$ may be assumed to be of the form $X^H X = (0) \oplus C_1$, $H = (\varepsilon) \oplus H_1$, where C_1 has no negative or zero eigenvalues. We shall assume that $\varepsilon = 1$; the case where $\varepsilon = -1$ is treated in the same way. Also, with respect to the same basis $A = (0) \oplus A_1$, where $X = UA$ is any H-polar decomposition of

X. We assume that A_1 is chosen in such a way that all its eigenvalues are in the open right half plane. Observe that, once A is fixed there are just two possibilities for choosing U. Indeed, both $\operatorname{Im} X$ and $\operatorname{Im} A$ are H-nondegenerate, and by Theorem 2.6 of [7] (with $p = 1$ and $q = 0$ in the notation of that theorem) there are just two H-unitary Witt extensions, once an isometry mapping $\operatorname{Im} A$ onto $\operatorname{Im} X$ is fixed. These two are described as follows. Let U_0 be the isometry from $\operatorname{Im} A$ onto $\operatorname{Im} X$. Let v be a vector such that $\operatorname{span}\{v\} = (H(\operatorname{Im} X))^{\perp}$. Since $\operatorname{Im} X$ is H-nondegenerate we have that either $\langle Hv, v\rangle > 0$ or $\langle Hv, v\rangle < 0$. Counting the number of negative and positive squares of H, we see that $\langle Hv, v\rangle$ must be positive. So, we may as well take v in such a way that $\langle Hv, v\rangle = 1$. Then $U_{\pm}$ defined by $U_{\pm}Ae_j = U_0Ae_j = Xe_j$ for $j \geqslant 2$, and $U_{\pm}e_1 = \pm v$ are the two Witt extensions of U_0 (the vectors $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$). We select the one with $\det U_{\pm} = 1$, and denote it in the sequel by U.

Let Y be an arbitrary perturbation of X. Then $Y^H Y$ is close to $X^H X$. Thus $Y^H Y$ has a unique eigenvalue λ_0 close to zero, and the corresponding eigenvector x_0 is close to e_1. Then there is an invertible matrix S, close to I, such that

$$S^{-1}Y^H YS = \begin{pmatrix} \lambda_0 & 0 \\ 0 & D_1 \end{pmatrix}, \quad S^* HS = \begin{pmatrix} 1 & 0 \\ 0 & G_1 \end{pmatrix}$$

with D_1 close to C_1 and G_1 close to H_1. As C_1 is a real matrix which is invertible and has no negative eigenvalues, it has a positive determinant. Hence also $\det D_1$ is positive. Since $A_1^2 = C_1$, we can apply the first part of Theorem 3.3 (see also the remark at the end of Section 3): there is a B_1 which is G_1-self-adjoint, close to A_1 and satisfies $B_1^2 = D_1$. As $\det Y^H Y \geqslant 0$ (by Theorem 2.1) and $\det D_1$ is nonnegative as well, it follows that λ_0 is nonnegative. Suppose first that λ_0 is positive. Then put

$$B = S\begin{pmatrix} \pm\sqrt{\lambda_0} & 0 \\ 0 & B_1 \end{pmatrix}S^{-1},$$

where the sign $\pm$ is to be determined later. Then B is H-self-adjoint and close to A and $B^2 = Y^H Y$, as one readily checks. Moreover, $\operatorname{Ker} B = \operatorname{Ker} Y$. So $Y = VB$ for some H-unitary matrix V. We select the sign in the definition of B in such a way that $\det V = 1$. If $\lambda_0 = 0$, we take B in the same way as above. In this case, as was seen in the previous paragraph, there are two Witt extensions of the isometry from $\operatorname{Im} B$ to $\operatorname{Im} Y$. We fix V to be the Witt extension with determinant 1. Obviously, we still have to show that $\|U - V\|$ is small.

Take

$$x \in \operatorname{Im}\left(S\begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix}S^{-1}\right).$$

Then $x = By$ for a unique $y \in \operatorname{span}\{Se_2, \dots, Se_n\}$, and $\|y\| \leqslant C\|B_1^{-1}\|\|x\|$, where $C = \|S\| \cdot \|S^{-1}\|$. With this we have

$$\|Ux - Vx\| = \|UBy - VBy\| \leqslant \|U(B-A)y\| + \|X - Y\|\|y\|$$
$$\leqslant C(\|U\|\,\|B-A\|\,\|B_1^{-1}\| + \|X-Y\|\,\|B_1^{-1}\|)\|x\|.$$

Now as B_1 is close to A_1 we have that $\|B_1^{-1}\|$ is uniformly bounded provided $\|X - Y\|$ is small enough. So $U - V$ is small in norm on the $(n-1)$-dimensional H-nondegenerate subspace

$$\mathscr{M} = \operatorname{Im}\left(S\begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix}S^{-1}\right);$$

more precisely, $\|(U|\mathscr{M}) - (V|\mathscr{M})\| \leqslant C_1\|X - Y\|$, where the constant $C_1 > 0$ is independent of Y.

Now recall that U and V are real H-unitary, and we have chosen both det U and det V equal to 1. These conditions determine U and V completely, provided $U|\mathscr{M}$ and $V|\mathscr{M}$ are known. It follows that $U - V$ is small in norm on the whole space. □

The proof of Theorem 5.2 shows that, under the hypotheses of this theorem, an H-polar decomposition $X = UA$ is H-stable provided A and $-A$ have no common nonzero eigenvalues (cf. the remark at the end of Section 3). Moreover, every such H-polar decomposition is *Lipschitz H-stable*, i.e., every Y sufficiently close to X admits an H-polar decomposition $Y = VB$ with $\|U - V\| + \|A - B\| \leqslant C\|X - Y\|$, where the positive constant C is independent of Y.

6. Analytic behavior

We conclude the paper with a result concerning the analytic behavior of H-polar decompositions when $F = \mathbb{C}$. Let $\mathscr{U}$ denote the set of H-unitary matrices, and let $\mathscr{A}$ denote the set of H-self-adjoint matrices. Let $\mathscr{SPD}$ denote the set of $n \times n$ matrices X such that $\sigma(X^H X) \cap (-\infty, 0] = \emptyset$. Observe that $\mathscr{SPD}$ is precisely the set of matrices that allow an H-stable H-polar decomposition.

Theorem 6.1. *Let $\Omega \in \mathbb{R}^m$ be an open set, and let $G_1 : \Omega \to \mathscr{SPD}$ be a real analytic function on Ω. Fix $\omega_0 \in \Omega$, and let $G_1(\omega_0) = U_0A_0$ be an H-polar decomposition with the property that A_0 has no negative eigenvalues* (*such an H-polar decomposition exists by Theorem* 4.1). *Then there exist real analytic functions $G_2 : \Omega \to \mathscr{U}$, $G_3 : \Omega \to \mathscr{A}$, such that $G_1(\omega) = G_2(\omega)G_3(\omega)$ is an H-polar decomposition for every $\omega \in \Omega$, and $G_2(\omega_0) = U_0$, $G_3(\omega_0) = A_0$.*

Proof. As $\sigma(X^H X) \cap (-\infty, 0] = \emptyset$ for every matrix $X \in \mathscr{SPD}$, the H-self-adjoint square root of $X^H X$ can be taken to depend analytically on X, as indicated in the proof of Theorem 3.3. Thus the H-self-adjoint factor in an H-polar decomposition can be chosen to depend analytically on the matrix X. Therefore also the H-unitary factor (which is uniquely determined once the H-self-adjoint factor is chosen) depends analytically on the matrix X. □

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