Block Cholesky Factorization of Infinite Matrices and Orthonormalization of Vectors of Functions

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Abstract. The following results on the block Cholesky factorization of bi-infinite and semi-infinite matrices are obtained. A method is proposed for computing the $LDM^T$- and block Cholesky factors of a bi-infinite banded block Toeplitz matrix. An equivalence relation is introduced to describe when two semi-infinite matrices with entries $A_{ij}$ coincide exponentially as $i, j, i + j \to \infty$. If two equivalent bi-infinite matrices have block Cholesky factorizations, then their block Cholesky factors and their inverses are equivalent. If a bi-infinite block matrix $A$ has a block Cholesky factorization whose lower triangular factor $L$ and its lower triangular inverse decay exponentially away from the diagonal, then the semi-infinite truncation of $A$ has a lower triangular block Cholesky factor whose elements approach those of $L$ exponentially. These results are then applied to studying the asymptotic behavior of vectors of functions obtained by orthonormalizing a large finite set of integer translates of an exponentially decaying vector of functions.

§1. Introduction

In a recent paper [11], several results on the Cholesky factorization of Gram matrices were obtained and applied to the study of the asymptotic behavior of splines obtained by orthonormalizing a large finite set of B-splines, in particular identifying the limiting profile when the knots are equally spaced. Additional properties of the Cholesky factorization of bi-infinite and semi-infinite matrices were obtained in [12] and applied to the study of the limiting profile arising from the orthonormalization of positive integer translates of an exponentially decaying function. In a previous paper [13] some results on the block Cholesky factorization of bi-infinite and semi-infinite Toeplitz matrices were obtained, which, in particular, give a method for computing it in the bi-infinite block tridiagonal case.

The purpose of this paper is to generalize the results on the Cholesky factorization of bi-infinite and semi-infinite matrices obtained in [11, 12] to the block matrix case, and to apply them to the study of the asymptotic behavior of vectors of functions obtained by orthonormalizing positive integer translates of a vector of exponentially decaying functions.

The paper is organized as follows. In Section 2, after a short review of results pertaining to the block $LDU$-factorization of real bi-infinite block
Toeplitz matrices, we generalize Theorem 5.1 of [13] on the block LDU-factorization of bi-infinite block tridiagonal matrices to the banded case. In Section 3, we extend the results of [11, 12] on the Cholesky factorization of bi-infinite and semi-infinite matrices to the block matrix case. In Section 4, we generalize some of the results derived in [12] on the limiting profile of functions obtained by orthonormalizing positive integer translates of an exponentially decaying function to vectors of exponentially decaying functions. In Section 5 we apply this result to a specific example. A crucial ancillary result on the stability of the block Cholesky factors of a positive definite real symmetric matrix perturbed by a matrix small in the Frobenius norm, is proved in the Appendix. This result, which is of independent interest, generalizes a previous result of Sun [16] but has been proved in an entirely different way.

§2. \( LDM^T \) Factorization of Banded Block Toeplitz Matrices

Let us first review some results on the block Cholesky factorization of real bi-infinite Toeplitz matrices of the form \((G_{i-j})_{i,j \in \mathbb{Z}}\) where each entry \(G_{i-j}\) is a square matrix of order \(k\) and \(Z\) is the set of all integers. A matrix may be viewed as a bounded linear operator on the Hilbert space \(\ell_2(\mathbb{Z})\) of square summable sequences indexed by the integers if and only if its so-called symbol

\[
\hat{G}(z) = \sum_{i=-\infty}^{\infty} z^i G_i, \quad |z| = 1,
\]

is essentially bounded, i.e., if all of its entries belong to \(L_\infty(\mathbb{T})\) where \(\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}\). In particular, if

\[
\sum_{i=-\infty}^{\infty} \|G_i\| < +\infty,
\]

where any matrix norm can be employed, then \(\hat{G}(z)\) is continuous on \(\mathbb{T}\) and the bi-infinite Toeplitz matrix \((G_{i-j})_{i,j \in \mathbb{Z}}\) is bounded on \(\ell_2(\mathbb{Z})\). The class of matrix functions \(\hat{G}(z)\) on \(\mathbb{T}\) of the form (2.1), where the coefficients \(G_i\) satisfy (2.2), is a Banach algebra with respect to the norm \(\|\hat{G}\| := \sum_{i=-\infty}^{\infty} \|G_i\|\), called the Wiener algebra of order \(k\).

Let \(G = (G_{i-j})_{i,j \in \mathbb{Z}}\) be a real block Toeplitz matrix satisfying (2.2) where each entry \(G_{i-j}\) of \(G\) is a square matrix of order \(k\). Then by an LDU-factorization of \(G\) we mean a representation of \(G\) of the form

\[
G = LDM^T,
\]

where the superscript \(T\) denotes matrix transposition and \(L = (L_{i-j})_{i,j \in \mathbb{Z}}\), \(M = (M_{i-j})_{i,j \in \mathbb{Z}}\) and \(D = (D_{i-j})_{i,j \in \mathbb{Z}}\) are block Toeplitz matrices having the following properties:
(a) \( L_0 = M_0 = I_k \) (the \( k \times k \) identity matrix),
(b) \( D_s = 0 \) for \( s \neq 0 \) and \( L_s = M_s = 0 \) for \( s < 0 \),
(c) the inverses \( L^{-1} \) and \( M^{-1} \) of \( L \) and \( M \) are block Toeplitz matrices satisfying \( [L^{-1}]_s = [M^{-1}]_s = 0 \) for \( s < 0 \),
(d) the series \( \sum_{s=1}^{\infty} \|L_s\|, \sum_{s=1}^{\infty} \|M_s\|, \sum_{s=1}^{\infty} \|L^{-1}_s\|, \) and \( \sum_{s=1}^{\infty} \|[M^{-1}]_s\| \) are convergent.

Passing to the symbols \( \widehat{L}(z), \widehat{D}(z) \equiv D_0 \) and \( \widehat{M}(z) \) of \( L, D \) and \( M \), one gets

\[
\widehat{G}(z) = \widehat{L}(z)D_0\widehat{M}(z^{-1})^T, \quad |z| = 1, \tag{2.4}
\]

where \( \widehat{L}(z) \) and \( \widehat{M}(z) \) extend to matrix functions that are continuous on the closed unit disk \( \{z \in \mathbb{C} : |z| \leq 1\} \) and analytic on the open unit disk. Further, \( \widehat{L}(z) \) and \( \widehat{M}(z) \) are nonsingular matrices for \( |z| \leq 1 \) and \( \widehat{L}(0) = \widehat{M}(0) = I_k \).

The factorization problems (2.3) and (2.4) for block Toeplitz matrices \( G \) satisfying (2.2) and their symbols \( \widehat{G}(z) \) in the Wiener algebra are equivalent [6].

Note that the factors in (2.3) and (2.4) are uniquely determined by \( G \) (or by \( \widehat{G}(z) \)). When \( G \) is a positive definite real symmetric matrix satisfying (2.2) in the sense that, for some \( \kappa > 0 \), \( x^T G x \geq \kappa x^T x \) for any real sequence \( (x_i)_{i \in \mathbb{Z}} \) in \( \ell_2(\mathbb{Z}) \), it has a unique \( LDU \)-factorization of the form (2.3) where \( L = M \) and \( D_0 \) is a positive definite real symmetric \( k \times k \) matrix [3]. Putting \( L_s = L_s D_0^{1/2} \), we then obtain the block Cholesky factorization

\[
G = LL^T, \tag{2.5}
\]

where \( L = (L_{i-j})_{i,j \in \mathbb{Z}} \) is invertible on \( \ell_2(\mathbb{Z}) \) with inverse \( L^{-1} \), the series \( \sum_{s=0}^{\infty} \|L_s\| \) and \( \sum_{s=0}^{\infty} \|[L^{-1}]_s\| \) converge, and \( L_0 \) is positive definite real symmetric.

Now let \( G \) be a bi-infinite banded block Toeplitz matrix \( G = (G_{i-j})_{i,j \in \mathbb{Z}} \), where \( G_{-m}, G_{-m+1}, \ldots, G_n \) are \( k \times k \) matrices and \( G_s = 0 \) for \( s < -m \) and \( s > n \). We consider the factorization (2.3) where \( L = (L_{i-j})_{i,j \in \mathbb{Z}}, L = (M_{i-j})_{i,j \in \mathbb{Z}} \) and \( D = (D_{i-j})_{i,j \in \mathbb{Z}} \) are banded block Toeplitz matrices with \( L_0 = M_0 = I_k \) (the \( k \times k \) identity matrix), \( L_s = -L_s^T \) \( (s = 1, \ldots, n) \) and \( M_s = -M_s^T \) \( (s = 1, 2, \ldots, m) \), \( D_0 = D, D_s = 0 \) for \( s \neq 0 \), \( L_s = 0 \) for \( s \neq 0, 1, \ldots, n \), and \( M_s = 0 \) for \( s \neq 0, 1, \ldots, m \).

The matrices \( L, M^T \) and \( D \) are factors in the factorization (2.3) if and only if \( L_1, \ldots, L_n, D, M_1, \ldots, M_m \) satisfy the relations

\[
\begin{align*}
G_0 &= D + L_1^T D M_1 + \cdots + L_{\min(m,n)}^T D M_{\min(m,n)}, \\
G_s &= -L_s^T D + L_{s+1}^T D M_1 + \cdots + L_{\min(n,m+s)}^T D M_{\min(n-s,m)}, \\
G_{-s} &= -D M_s + L_1^T D M_{s+1} + \cdots + L_{\min(m-s,n)}^T D M_{\min(m,n+s)}, \\
G_n &= -L_n^T D, \\
G_{-m} &= -D M_m.
\end{align*}
\]
This factorization, which exists under very general hypotheses, is not unique. We are especially interested in giving an effective method for finding the unique factorization such that $L$ and $M$ are boundedly invertible on the Hilbert space $\ell_2(\mathbb{Z})$ and, further, $L^{-1}$ and $M^{-1}$ are lower triangular block Toeplitz matrices, i.e., such that (2.3) is an $LDU$-factorization.

Such a method is given by (the proof of) Theorem 2.1, which generalizes a previous result for tridiagonal block Toeplitz matrices [13]. In formulating this result significant use is made of the theory of monic matrix polynomials [7, 8, 14].

Let $G$ be the bi-infinite banded block Toeplitz matrix considered above and $G = LDM^T$ its factorization (2.3). We shall assume that $G_n$ is nonsingular. Similar results hold if one assumes the invertibility of $G_{-m}$. Now let the real matrix function
\[
\Sigma(z) = \sum_{j=-m}^{n} z^j G_j,
\] (2.6)
be the symbol of the matrix $G$. Then $P(z) = z^m G_n^{-1} \Sigma(z)$ is a monic matrix polynomial of degree $m + n$, "monic" meaning that the coefficient of $z^{m+n}$ is the $k \times k$ unit matrix. Now let $\Gamma$, with $0 \notin \Gamma$, be a simple closed oriented rectifiable Jordan curve dividing the complex plane into an interior domain $\Omega_+$ with $0 \in \Omega_+$ and an exterior domain $\Omega_-$ with $\infty \in \Omega_-$, and assume that $\det(z^m \Sigma(z))$ does not vanish for $z \in \Gamma$. Then (2.3) is valid if and only if
\[
\Sigma(z) = \left( I_k - \sum_{i=1}^{n} z^i L_i \right)^T D \left( I_k - \sum_{j=1}^{m} z^{-j} M_j \right).
\] (2.7)

Moreover, if $G_n$ is nonsingular, it follows from $G_n = -L_n^T D$ that both $L_n$ and $D$ are nonsingular.

Before discussing the main theorem, we review some well-known notions on eigenvalues and Jordan chains [10, 14].

Let $W(z)$ be a $k \times k$ matrix polynomial, i.e., a polynomial in $z$ whose coefficients are real or complex $k \times k$ matrices such that $\det W(z)$ is a scalar polynomial in $z$. We call $z_0 \in \mathbb{C}$ an eigenvalue of $W$ if $\det W(z_0) = 0$, and $\varphi_0 \in \mathbb{C}^k$ an eigenvector of $W$ corresponding to the eigenvalue $z_0$ if $\varphi_0 \neq 0$ and $W(z_0) \varphi_0 = 0$. The set of eigenvalues of $W$ is called the spectrum of $W$; it coincides with the finite set of zeros of the scalar polynomial $\det W(z)$.

Let $z_0$ be an eigenvalue of $W$. Then the $u$-uple of vectors $\varphi_0, \varphi_1, \ldots, \varphi_{u-1}$ in $\mathbb{C}^k$ is called a Jordan chain at $z_0$ of length $u$ if $\varphi_0 \neq 0$ and the lower triangular linear system of equations
\[
\sum_{j=0}^{p} \frac{W^{(p-j)}(z_0)}{(p-j)!} \varphi_j = 0, \quad p = 0, 1, \ldots, u - 1,
\] (2.8)
is valid. The $ku \times ku$ matrix of the linear system (2.8) will be denoted by $J_W^{(u)}(z_0)$. If one considers the subspace of $\mathbb{C}^k$ spanned by the vectors constituting the Jordan chains at the eigenvalue $z_0$ and orders the linearly independent
maximal Jordan chains, say, according to decreasing length until one has a basis of this subspace, the lengths of these chains are the so-called \textit{partial multiplicities} of $W$ at the eigenvalue $z_0$. They do not depend on the specific choice of the Jordan chains but only on $W$. The sum of the partial multiplicities at the eigenvalue $z_0$ coincides with the order of $z_0$ as a zero of $\det W(z)$ and is called the \textit{algebraic multiplicity} of $z_0$; the dimension of the kernel of $W(z_0)$, which coincides with the number of linearly independent maximal Jordan chains at $z_0$, is called its \textit{geometric multiplicity}.

Let us introduce some basic facts about matrix polynomials [7, 8, 14]. Consider the monic matrix polynomial

$$
P(z) = z^t I_k + z^{t-1} A_{t-1} + \cdots + zA_1 + A_0,
$$

where the coefficients are $k \times k$ matrices. Since $\det P(z)$ is a scalar polynomial of degree $k\ell$, there are exactly $k\ell$ eigenvalues, when counted according to algebraic multiplicity. Let $z_1, \ldots, z_p$ be the \textit{distinct} eigenvalues of $P(z)$. For each $j$, we construct a pair of matrices $X_j$ (of size $k \times m_j$ where $m_j$ is the multiplicity of $z_j$ as a zero of $\det P(z)$) and $T_j$ (of size $m_j \times m_j$) as follows:

$$
X_j = \begin{bmatrix} x_{11}^{(j)} & \cdots & x_{1r_1}^{(j)} & x_{21}^{(j)} & \cdots & x_{2r_2}^{(j)} & \cdots & x_{q1}^{(j)} & \cdots & x_{qr_q}^{(j)} \end{bmatrix};
$$

$$
T_j = \text{diag} \left( J_{r_1}(z_j), J_{r_2}(z_j), \ldots, J_{r_q}(z_j) \right).
$$

Here $x_{s1}^{(j)}, x_{s2}^{(j)}, \ldots, x_{sr_s}^{(j)}$ for $s = 1, \ldots, q$ are the maximal Jordan chains for $P(z)$ corresponding to $z_j$ such that the vectors $x_{11}^{(j)}, x_{21}^{(j)}, \ldots, x_{q1}^{(j)}$ are linearly independent and $r_1 + r_2 + \cdots + r_q = m_j$, and $J_{r_s}(z_j)$ is the $r_s \times r_s$ upper triangular Jordan block with eigenvalue $z_j$. Finally, put

$$
X = \begin{bmatrix} X_1^{} & X_2^{} & \cdots & X_p^{} \end{bmatrix}, \quad T = \text{diag} \left( T_1, T_2, \ldots, T_p \right).
$$

One can show [7, 8] that the $k\ell \times k\ell$ matrix

$$
col [XT^j]_{j=0}^{t-1} := \begin{bmatrix} X^{} & XT^{} & \cdots & XT^{t-1} \end{bmatrix}
$$

is invertible and that

$$
XT^t + \sum_{j=0}^{t-1} A_j XT^j = 0.
$$

More generally, let $P(z)$ be the monic $k \times k$ matrix polynomial given by (2.9). Then the pair of matrices $(X, T)$, where $X$ is of size $k \times k\ell$ and $T$ is of size $k\ell \times k\ell$, is called a \textit{right spectral pair} for the polynomial $P(z)$ if the operator in (2.12) is invertible and (2.13) holds. Thus the pair $(X, T)$ given by (2.11) is a right spectral pair for $P(z)$. According to [14], Theorem 2.1.1, for
every couple of right spectral pairs \((X_1, T_1)\) and \((X_2, T_2)\) of the same monic matrix polynomial there exists a unique invertible matrix \(S\) such that
\[
X_1 = X_2S, \quad T_1 = S^{-1}T_2S.
\]

According to [14], Theorem 2.2.1, we have the following: If \((X, T)\) is a right spectral pair of the monic matrix polynomial \(P(z)\) given by (2.9), then
\[
P(z) = z^t - XT^t(V_1 + zV_2 + \cdots + z^{t-1}V_t)
\]
where \(V_1, \ldots, V_t\) are the \(k \ell \times k\) matrices defined by
\[
[V_1 \quad V_2 \quad \cdots \quad V_t] = (\text{col}[XT^j]_{j=0}^{t-1})^{-1}.
\]

This representation is called a right canonical form of \(P(z)\). Moreover,
\[
P(z)^{-1} = X(z - T)^{-1}V_t, \quad |z| > \|T\|.
\]

Conversely, if there exist a \(k \times k\ell\) matrix \(X\), a \(k \ell \times k\ell\) matrix \(T\) and a \(k \ell \times k\) matrix \(V_t\) such that (2.16) holds, then \((X, T)\) is a right spectral pair for \(P(z)\) and (2.14) and (2.15) are valid.

We now discuss the main theorem of this section. Let \(\lambda_1, \ldots, \lambda_s\) be the distinct zeros of \(\det(z^n \Sigma(z))\) in \(\Omega_-\), let \(\mu_1, \ldots, \mu_t\) be the distinct zeros of \(\det(z^n \Sigma(z))\) in \(\Omega_+\), put \(z_j = \lambda_j\) for \(j = 1, \ldots, s\) and \(z_j = \mu_{j-s}\) for \(j = s + 1, \ldots, s + t\), and let \(m_j, j = 1, \ldots, s + t\) denote the multiplicity of \(z_j\) as a zero of \(\det(z^n \Sigma(z))\). Then \(m_1 + \cdots + m_{s+t} = (m + n)k\). The main result of this section is a theorem which gives a constructive proof of the existence of the factorization (2.3).

**THEOREM 2.1.** Let \((X, T)\) be a right spectral pair of the monic matrix polynomial \(P(z) = z^nG_{-1}^{m} \Sigma(z)\). Then there exists a factorization of \(\Sigma(z)\) of the type (2.7) where \(\det(I_k - \sum_{k=1}^{m} z^k \mathcal{L}_k) \neq 0\) for \(z \in \Omega_+\) and \(\det(I_k - \sum_{k=1}^{m} z^{-k} \mathcal{M}_k) \neq 0\) for \(z \in \Omega_-\), if and only if
\[
m_1 + \cdots + m_s = nk, \quad m_{s+1} + \cdots + m_{s+t} = mk,
\]
and the restriction of \(\text{col}[XT^j]_{j=0}^{m-1}\) to the linear span of the eigenvectors and generalized eigenvectors of \(T\) in \(\Omega_+\) is invertible. If this factorization exists, it is unique and is called the spectral factorization of \(G\).

This theorem and its proof may be derived directly from the theory of monic matrix polynomials [7, 8, 14]. Here we give a full proof generalizing the analogous result for symbols of tridiagonal block Toeplitz matrices [13], where the supplementary condition on \(\text{col}[XT^j]_{j=0}^{m-1}\) is always satisfied.

**PROOF.** As \(D\) is nonsingular, if \(\Sigma(z)\) has the factorization (2.7), then by passing to the determinant one sees that (2.17) is satisfied.
Now suppose the relation (2.17) is true. As in (2.10), we construct the right spectral pair \((X_r, T_r)\) of \(\mathcal{P}(z)\) by using its Jordan chains (which are also the Jordan chains of \(z^n\Sigma(z)\)) and the right spectral pair \((X_l, T_l)\) of \(\mathcal{P}(z)^T\) using its Jordan chains (which are also the Jordan chains of \(z^n\Sigma(z)^T(G_n^T)^{-1}\)), where the distinct eigenvalues \(z_1, \ldots, z_{s+t}\) of \(z^n\Sigma(z)\) have been ordered by decreasing modulus. Then, denoting the \(q \times q\) upper triangular Jordan block with diagonal entries \(\lambda\) by \(J_q(\lambda)\) and partitioning the \(k \times (m + n)k\) matrices \(X_l\) and \(X_r\) into a \(k \times nk\) block and a \(k \times mk\) block, respectively, we have

\[
X_l = \begin{bmatrix} V & ? \end{bmatrix}, \quad X_r = \begin{bmatrix} ? & W \end{bmatrix},
\]

\[
T_l = T_r = \text{diag} \left( J_{m_1}(\lambda_1), \ldots, J_{m_s}(\lambda_s), J_{m_{s+1}}(\mu_1), \ldots, J_{m_{s+t}}(\mu_t) \right),
\]

where \(V\) is a \(k \times nk\) matrix, \(W\) is a \(k \times mk\) matrix and the \(k \times mk\) and \(k \times nk\) matrices at the question marks are irrelevant. Write

\[
\Lambda_L^{-1} = \text{diag} \left( J_{m_1}(\lambda_1), \ldots, J_{m_s}(\lambda_s) \right),
\]

\[
\Lambda_M = \text{diag} \left( J_{m_{s+1}}(\mu_1), \ldots, J_{m_{s+t}}(\mu_t) \right),
\]

and construct the \(mk \times k\) matrices \(V_1, \ldots, V_m\) and the \(nk \times k\) matrices \(W_1, \ldots, W_n\) by putting

\[
\begin{bmatrix} V_1 & V_2 & \cdots & V_m \end{bmatrix} = \left( \text{col} \left[ W \Lambda_M^j \right]_{j=0}^{m-1} \right)^{-1} ;
\]

\[
\begin{bmatrix} W_1 & W_2 & \cdots & W_n \end{bmatrix} = \left( \text{col} \left[ V \Lambda_L^{-j} \right]_{j=0}^{n-1} \right)^{-1},
\]

where the first column matrix is assumed to be nonsingular and the second column matrix is nonsingular as a result. Then we obtain the factorization

\[
\mathcal{P}(z) = z^n G_n^{-1} \Sigma(z) = G_n^{-1} \left( I_k - \sum_{i=1}^{n} z^i \mathcal{L}_i^T \right) D \left( z^n I_k - \sum_{j=1}^{m} z^{m-j} \mathcal{M}_j \right)
\]

by defining

\[
\mathcal{L}_i = -(G_n^T)^{-1} \left[ V \Lambda_L^{-n} W_i \right]^{-1} V \Lambda_L^{-n} W_{i+1} G_n^T, \quad i = 1, \ldots, n - 1;
\]

\[
\mathcal{L}_n = (G_n^T)^{-1} \left[ V \Lambda_L^{-n} W_i \right]^{-1} G_n^T, \quad D = -(\mathcal{L}_n^T)^{-1} G_n;
\]

\[
\mathcal{M}_i = W \Lambda_M^m V_{m+1-i}, \quad i = 1, \ldots, m.
\]

The expressions for \(\mathcal{L}_i\), \(D\) and \(\mathcal{M}_i\) are easily found from the requirement that \((W, \Lambda_M)\) is a right spectral pair for the polynomial \(M(z)\) and \((V, \Lambda_L^{-1})\) is a right spectral pair for the polynomial \(L(z)\) given by

\[
M(z) = z^n I_k - \sum_{s=0}^{m-1} z^s \mathcal{M}_{m-s}, \quad L(z) = D^T \left( I_k - \sum_{i=1}^{n} z^i \mathcal{L}_i \right) (G_n^T)^{-1}.
\]

(2.19)
Indeed, put
\[
\begin{align*}
\hat{G}_0 &= D + \mathcal{L}_1^T DM_1 + \cdots + \mathcal{L}_{\min(m,n)}^T DM_{\min(m,n)}, \\
\hat{G}_s &= -\mathcal{L}_s^T D + \mathcal{L}_{s+1}^T DM_1 + \cdots + \mathcal{L}_{\min(n,m+s)}^T DM_{\min(n-s,m)}, \\
&\quad s = 1, \ldots, n-1 \\
\hat{G}_{s-m} &= -D M_s + \mathcal{L}_1^T DM_{s+1} + \cdots + \mathcal{L}_{\min(m-s,n)}^T DM_{\min(m+s)}, \\
&\quad s = 1, \ldots, m-1 \\
\hat{G}_n &= -\mathcal{L}_n^T D = G_n, \quad \hat{G}_{-m} = -D M_m.
\end{align*}
\]
Then for \( z \in \mathbb{C} \setminus \{0\} \) we have
\[
\hat{\Sigma}(z) := \left( I_k - \sum_{i=1}^n z^i \mathcal{L}_i^T \right) D \left( I_k - \sum_{j=1}^m z^{-j} M_j \right) = \sum_{j=-m}^n z^j \hat{G}_j.
\]
Now let \( w_0, w_1, \ldots, w_{u-1} \) be the Jordan chain of \( z^m \hat{\Sigma}(z) \) corresponding to the eigenvalue \( \mu_j \) occupying the \( N \)-th through the \( (N+u-1) \)-th columns of the matrix \( W \), and let \( (e_j)_i = \delta_{i,j} \) \((i = 1, \ldots, mk; j = N, \ldots, N+u-1) \). Then for \( p = 0, 1, \ldots, u - 1 \) we have
\[
\sum_{t=0}^p \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t z^m \hat{\Sigma}(z) \right]_{z=\mu_j} w_{p-t} = 0.
\]
Moreover, for \( p = 0, 1, \ldots, u - 1 \) we have
\[
\sum_{t=0}^p \binom{m}{t} \mu_j^{m-t} w_{p-t} = W \Lambda^m_{M} e_{N+p} = \sum_{s=0}^{m-1} M_{m-s} W \Lambda^s_{M} e_{N+p}
\]
\[
= \sum_{s=0}^{m-1} \sum_{t=0}^{\min(p,s)} \binom{s}{t} \mu_j^{s-t} w_{p-t} = \sum_{t=0}^{p} \sum_{s=t}^{m-1} \binom{s}{t} \mu_j^{s-t} M_{m-s} w_{p-t},
\]
which can be written as
\[
\sum_{t=0}^p \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t M(z) \right]_{z=\mu_j} w_{p-t} = 0, \quad p = 0, 1, \ldots, u - 1,
\]
where \( M(z) \) is given by (2.19). As, by construction, \( z^m \hat{\Sigma}(z) = G_n L(z)^T M(z) \) where \( L(z) \) and \( M(z) \) are the monic matrix polynomials defined by (2.19), we have
\[
\sum_{t=0}^p \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t z^m \hat{\Sigma}(z) \right]_{z=\mu_j} w_{p-t}
\]
\[
= G_n \sum_{\rho=0}^p \frac{1}{\rho!} \sum_{t=0}^p \frac{1}{(t-\rho)!} \left[ \left( \frac{d}{dz} \right)^{t-\rho} L(z)^T \right]_{z=\mu_j} \left[ \left( \frac{d}{dz} \right)^\rho M(z) \right]_{z=\mu_j} w_{p-t}
\]
\[
= 0.
\]
Hence \( w_0, w_1, \ldots, w_{u-1} \) is a Jordan chain of \( z^m \hat{\Sigma}(z) \) corresponding to the eigenvalue \( \mu_j \).

Next, let \( v_0, v_1, \ldots, v_{u-1} \) be the Jordan chain of \( z^m \Sigma(z)^T (G_n^T)^{-1} \) corresponding to the eigenvalue \( \lambda_j \) occupying the \( N \)-th through the \( (N + u - 1) \)-th columns of the matrix \( V \), and let \( (e_j)_i = \delta_{i,j} \) (\( i = 1, \ldots, nk; \ j = N, \ldots, N + u - 1 \)). Then for \( p = 0, 1, \ldots, u - 1 \) we have

\[
\sum_{t=0}^{p} \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t z^m \Sigma(z)^T (G_n^T)^{-1} \right]_{z=\lambda_j} v_{p-t} = 0.
\]

Using the identity

\[
V L^{-n} = V L^{-n} W_1 V + \sum_{i=1}^{n-1} V L^{-n} W_{i+1} V L^{-i}
\]

\[
= V L^{-n} W_1 \left\{ V - \sum_{i=1}^{n-1} \mathcal{L}_i V L^{-i} \right\},
\]

for \( p = 0, 1, \ldots, u - 1 \) we have

\[
\sum_{t=0}^{p} \binom{n}{t} \lambda_j^{n-t} v_{p-t} = V L^{-n} e_{N+p} = V L^{-n} W_1 \left\{ V - \sum_{i=1}^{n-1} \mathcal{L}_i V L^{-i} \right\} e_{N+p}
\]

\[
= V L^{-n} W_1 \left\{ v_p - \sum_{i=1}^{n-1} \mathcal{L}_i \sum_{t=0}^{\max(i,p)} \binom{i}{t} \lambda_j^{i-t} v_{p-t} \right\}
\]

\[
= V L^{-n} W_1 \left\{ \left[ I_k - \sum_{i=1}^{n-1} \lambda_j^i \mathcal{L}_i \right] v_p - \sum_{t=1}^{p} \sum_{i=1}^{n-1} \binom{i}{t} \lambda_j^{i-t} \mathcal{L}_i v_{p-t} \right\},
\]

which can also be written as

\[
\sum_{t=0}^{p} \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t L(z) \right]_{z=\lambda_j} v_{p-t} = 0.
\]

Hence \( v_0, v_1, \ldots, v_{u-1} \) is a Jordan chain of \( L(z) \) corresponding to the eigenvalue \( \lambda_j \). Since \( P(z)^T = M(z)^T L(z) \), \( v_0, v_1, \ldots, v_{u-1} \) is also a Jordan chain of \( P(z) \) and hence of \( z^m \hat{\Sigma}(z)^T (G_n^T)^{-1} \) corresponding to the eigenvalue \( \lambda_j \).

Now let \( Q_i = G_n^{-1} [G_{i-m} - G_{i-m}] \) \( (i = 0, 1, \ldots, m + n - 1) \) and \( Q(z) = z^m G_n^{-1} [\Sigma(z) - \hat{\Sigma}(z)] \). Then for any Jordan chain \( w_0, w_1, \ldots, w_{u-1} \) of \( P(z) \) corresponding to \( \mu_j \) we have for \( p = 0, 1, \ldots, u - 1 \)

\[
\sum_{t=0}^{p} \frac{1}{t!} \left[ \left( \frac{d}{dz} \right)^t Q(z) \right]_{z=\mu_j} = \sum_{i=0}^{m+n-1} Q_i \left[ w_{p-1} \cdots w_1 \ w_0 \right] J_p(\mu_j)^i = 0,
\]
and hence
\[
\sum_{i=0}^{m+n-1} Q_i W \Lambda_M^i = 0. \tag{2.20}
\]
In the same way we obtain
\[
\sum_{i=0}^{m+n-1} Q_i^T V \Lambda_L^{-i} = 0. \tag{2.21}
\]

From (2.20) and (2.21) we now get
\[
\left( \sum_{i=1}^{m+n-1} V^T Q_i W \Lambda_M^{i-1} \right) \Lambda_M = (\Lambda_L^T)^{-1} \left( \sum_{i=1}^{m+n-1} (\Lambda_L^T)^{-(i-1)} V^T Q_i W \right) = -V^T Q_0 W,
\]
which, in view of the disjointness of the spectra of \( \Lambda_M \) and \( (\Lambda_L^T)^{-1} \) (cf. [15]), implies that \( V^T Q_0 W \) and the two sums between brackets vanish. Repeating this argument we obtain
\[
V^T Q_0 W = V^T Q_1 W = \cdots = V^T Q_{m+n-1} W = 0.
\]

Choosing any set \( \{Q_{i_1}, \ldots, Q_{i_m}\} \) of \( m \) matrices \( Q_i \), we now easily find that
\[
V^T [Q_{i_1} \cdots Q_{i_m}] \col [W \Lambda_M^{i}]_{j=0}^{m-1} = \sum_{j=0}^{m-1} V^T Q_{i_{j-1}} W \Lambda_M^j = 0,
\]
implying
\[
V^T Q_0 = V^T Q_1 = \cdots = V^T Q_{m+n-1} = 0.
\]
Similarly, choosing any set \( \{Q_{i_1}, \ldots, Q_{i_n}\} \) of \( n \) matrices \( Q_i \), we obtain
\[
[Q_{i_1}^T \cdots Q_{i_n}^T] \col [V \Lambda_L^{-j}]_{j=0}^{n-1} = \sum_{j=0}^{n-1} Q_{i_{j-1}}^T V \Lambda_L^{-j} = 0,
\]
which implies \( Q_0 = Q_1 = \cdots = Q_{m+n-1} = 0 \). Consequently, the two monic matrix polynomials \( P(z) \) and \( \hat{P}(z) \) coincide. ■

Necessary and sufficient conditions for \( \Sigma(z) \) to have a factorization of the form (2.7) where \( \det(I_k - \sum_{k=1}^n z^k L_k) \neq 0 \) for \( z \in \Omega_+ \) and \( \det(I_k - \sum_{k=1}^n z^{-k} M_k) \neq 0 \) for \( z \in \Omega_- \), are well-known, even if the coefficient \( G_n \) in
(2.6) is singular ([9], Theorem 4.8). In fact, \( \Sigma(z) \) has such a factorization if and only if the ranks of the two finite block Toeplitz matrices
\[
\begin{bmatrix}
D_0 & \cdots & D_{-m-n+1} \\
\vdots & & \vdots \\
D_{m-1} & \cdots & D_{-n}
\end{bmatrix},
\begin{bmatrix}
D_{-1} & \cdots & D_{-m-n-1} \\
D_0 & \cdots & D_{-m-n} \\
\vdots & & \vdots \\
D_{m-1} & \cdots & D_{-n}
\end{bmatrix},
\]
are both equal to \( mk \). Here \( D_j = (2\pi i)^{-1} \int_\Gamma z^{-j-1} \Sigma(z)^{-1} \, dz \). However, this result, which is very powerful from the theoretical point of view, does not suggest an algorithm for computing the factors. A possible method for the case when both \( G_n \) and \( G_{-m} \) are singular, is to write \( z^m \Sigma(z) \) in the form
\[
z^m \Sigma(z) = \sum_{j=0}^{m+n} \sum_{r=j}^{m+n} \binom{r}{j} z_0^{-r-j} G_{r-m}(z - z_0)^j
\]
for some \( z_0 \neq 0 \) such that \( \Sigma(z_0) \) is nonsingular and to apply monic matrix polynomial factorization theory to (2.22) viewed as a function of \( (z - z_0)^{-1} \). When \( z_0 \) is sufficiently close to zero, the shift of the factorization contour will not affect the block Cholesky factors obtained.

When \( \Sigma(z) \) is the symbol of a positive definite real symmetric banded block Toeplitz matrix \( G = (G_{i-j})_{i,j \in \mathbb{Z}} \), then the unit circle can be assumed to be the curve mentioned above and \( G \) to have a unique LDU-factorization (2.1) with \( L = M \) and a positive definite real symmetric diagonal factor \( D \). Further, \( G_0 \) is positive definite real symmetric and \( G_{-s} = G_s^T \) (\( s = 1, \ldots, m = n \)). Assume \( G_m \) to be nonsingular. Then the monic matrix polynomial \( \mathcal{P}(z) = z^m G_m^{-1} \Sigma(z) \) has a factorization of the form
\[
\mathcal{P}(z) = L(z)^T M(z), \quad |z| = 1,
\]
where \( M(z) \) is a monic matrix polynomial of degree \( m \) with its eigenvalues in the open unit disk and \( L(z) \) is a monic matrix polynomial of degree \( m \) with its eigenvalues in the exterior region of the unit circle. Then (2.4) has the form (2.7) where \( \mathcal{M}_j = L_j \) (\( j = 1, \ldots, m \)) and \( D \) is positive definite real symmetric. It is then sufficient to compute the monic polynomial right factor \( M(z) \) of \( \mathcal{P}(z) \) using the identities (2.18) and \( \mathcal{M}_i = W A^n_M V_{m+1-i} \) (\( i = 1, \ldots, m \)), and to put \( D = -\mathcal{M}_m^{-1} G_m \).

§3. Block Cholesky Factorization of Positive Definite Matrices

In this section we collect some results on the asymptotic behavior of the block Cholesky factorization of large positive definite matrices. Let \( J \) be some nonempty subset of \( \mathbb{Z} \), the set of all integers, and \( \ell_2(J) \) the space of real square
summable sequences on $J$ with the usual norm and scalar product. We recall that a real symmetric matrix is called positive semi-definite if

$$x^T Ax \geq 0, \quad x \in \ell_2(J),$$

and positive definite if there exists a strictly positive constant $\kappa$ such that

$$x^T Ax \geq \kappa x^T x, \quad x \in \ell_2(J).$$

The matrix $A$ is bounded on $\ell_2(J)$ if

$$\|A\|_2 := \sup\{\|Ax\|_2 : \|x\|_2 = 1, \; x \in \ell_2(J)\} < +\infty.$$

In this section we are interested in results similar to those in [11], but using $k \times k$ matrices instead of scalars as elements of bi-infinite and semi-infinite matrices. By a bi-infinite block matrix $A$ of order $k$ we mean a matrix with entries defined on $J = Z$ where the entries $A_{ij}$ ($i, j \in Z$) are $k \times k$ matrices. By a semi-infinite block matrix of order $k$ we denote a matrix $A$ whose entries are $k \times k$ matrices defined on the set $J = Z_+$, the set of nonnegative integers. Bi-infinite and semi-infinite block matrices of order $k = 1$ will be called bi-infinite and semi-infinite matrices, respectively. We call a bi-infinite or semi-infinite block matrix $A$ of order $k$ lower block triangular (upper block triangular) if $A_{ij}$ is the zero $k \times k$ matrix whenever $i < j$ ($i > j$).

Every bi-infinite block matrix $A = (A_{ij})_{i,j \in Z}$ of order $k$ has a truncation to a semi-infinite block matrix given by

$$A_+ = (A_{ij})_{i,j \in Z_+}.$$

Similarly, to every semi-infinite block matrix $A = (A_{ij})_{i,j \in Z_+}$ there corresponds a sequence of finite truncations

$$A_N = (A_{ij})_{i,j = 0,1,\ldots,N}, \quad N = 0, 1, 2, \ldots,$$

to a matrix of order $(N + 1)k$.

In the present section we shall make extensive use of the following two definitions which are direct generalizations of notions introduced in [11, 12].

Let $A$ be a bi-infinite (semi-infinite) matrix whose entries are matrices of order $k$. Then $A$ is said to be exponentially decaying if, for some norm on the $k \times k$ matrices, there exist numbers $c > 0$ and $\lambda \in (0, 1)$ such that $\|A_{ij}\| \leq c\lambda^{i-j}$ for all $i, j \in Z$ ($i, j \in Z_+$ where $Z_+$ is the set of all nonnegative integers).

Let $A$ and $B$ be two bi-infinite or semi-infinite matrices whose entries are matrices of order $k$. We say that $A \sim B$ if, for some norm on the $k \times k$ matrices, there exist two numbers $c > 0$ and $\lambda \in (0, 1)$ such that

$$\|A_{ij} - B_{ij}\| \leq c\lambda^{i+j}, \quad i, j \geq 0.$$
Clearly, the equivalence relation $\sim$ does not depend on the choice of the matrix norm for the $k \times k$ matrices. The equivalence relation enables us to study the asymptotic behavior of infinite matrices as $i, j \geq 0$ and $i + j \to +\infty$.

To every semi-infinite block matrix $A$ of order $k$ we associate the following semi-infinite matrix $B$:

$$B_{p,q} = \left(A_{\left\lfloor \frac{p}{k}\right\rfloor, \left\lfloor \frac{q}{k}\right\rfloor}\right)_{p+1-k\left\lfloor \frac{p}{k}\right\rfloor, q+1-k\left\lfloor \frac{q}{k}\right\rfloor},$$

(3.1)

where one should recall that every entry of $A$ is a $k \times k$ matrix with rows and columns numbered from 1 to $k$ and where $[x]$ is the largest integer not exceeding $x$. In a similar way one can associate to every bi-infinite block matrix $A$ of order $k$ the bi-infinite matrix $B$ with entries $B_{p,q}$ given by (3.1) where $p, q \in \mathbb{Z}$.

**LEMMA 3.1.** An exponentially decaying matrix is a bounded operator on $\ell_2(J)$.

**PROOF.** We only give the proof for $J = \mathbb{Z}$. Choose $c > 0$ and $\lambda \in (0, 1)$ such that $\|A_{ij}\| \leq c\lambda^{\mid i-j\mid}$ for all $i, j \in \mathbb{Z}$. Then for any $i \in \mathbb{Z}$ and $x \in \ell_2(\mathbb{Z})$, where we write $x$ as an infinite column vector with column vectors of order $k$ as entries, we have the equation

$$(Ax)_i = \sum_{j = -\infty}^{\infty} A_{i,i+j}x_{i+j},$$

which implies

$$\|Ax\|_2 \leq \sum_{j = -\infty}^{\infty} \|\left(A_{i,i+j}x_{i+j}\right)_{i = -\infty}^{\infty}\|_2 \leq \sum_{j = -\infty}^{\infty} c\lambda^{\mid j\mid}\|x\|_2 \leq \frac{2c}{1-\lambda}\|x\|_2,$$

which completes the proof. $\blacksquare$

A different proof is to observe that $A$ is exponentially decaying if and only if the matrix $B$ constructed from it in (3.1) is exponentially decaying, and then to apply Lemma 3.1 of [11]. Because of the usefulness of this invariance property for exponentially decaying matrices, we make it into a separate result.

**LEMMA 3.2.** A bi-infinite or semi-infinite block matrix $A$ of order $k$ is exponentially decaying if and only if the bi-infinite or semi-infinite matrix $B$ given by (3.1) is exponentially decaying.

**PROOF.** We first observe that there exist constants $c_1, c_2 > 0$ depending on the $k \times k$ matrix norm such that for any matrix $C$ of order $k$

$$c_1 \sum_{r,s=1}^{k} |C_{rs}| \leq \|C\| \leq c_2 \sum_{r,s=1}^{k} |C_{rs}|.$$

(3.2)
Hence if \( A \) is a bi-infinite or semi-infinite block matrix of order \( k \) and \( c > 0 \) and \( \lambda \in (0, 1) \) are such that \( \|A_{ij}\| \leq c\lambda^{|i-j|} \) for \( i, j \in J \), then

\[
B_{p,q} \leq \frac{1}{c_1} \left\| A_{\{x\},\{y\}} \right\| \leq \frac{c}{c_1} \lambda |x-y| \leq \frac{c}{c_1} \lambda \sqrt{\lambda} \left( \sqrt{\lambda} \right)^{|p-q|},
\]

implying that \( B \) is exponentially decreasing. Conversely, let \( d > 0 \) and \( \mu \in (0, 1) \) be such that \( |B_{p,q}| \leq d\mu^{|p-q|} \) for \( p, q \in J \). Then

\[
\|A_{ij}\| \leq c_2 \sum_{p=ki}^{ki+k-1} \sum_{q=kj}^{kj+k-1} |B_{p,q}| \leq c_2 d \sum_{p=ki}^{ki+k-1} \sum_{q=kj}^{kj+k-1} \mu^{|p-q|} \]
\[
= c_2 d \sum_{r,s=0}^{k-1} \mu^{|k(i-j)+r-s|} \leq c_2 d k^2 (\mu^k)^{|i-j|},
\]

which implies that \( A \) is exponentially decreasing. \( \blacksquare \)

Given two bi-infinite (semi-infinite) block matrices \( A^{(1)} \) and \( A^{(2)} \) of order \( k \), we construct from them \( B^{(1)} \) and \( B^{(2)} \) according to (3.1). Since the conversion \( A \mapsto B \) is essentially a matter of book keeping, we see that the matrix corresponding to \( A^{(1)} \) \( A^{(2)} \) according to (3.1) is exactly \( B^{(1)} \) \( B^{(2)} \). Using Lemma 3.2 of [11] and the present Lemma 3.2 we immediately have

**Lemma 3.3.** A product of exponentially decaying bi-infinite or semi-infinite block matrices is exponentially decaying.

Recalling the equivalence relation between bi-infinite or semi-infinite matrices, it is easily seen that if \( A^{(1)} \) and \( A^{(2)} \) are semi-infinite block matrices, \( A^{(2)} \) decays exponentially and \( A^{(1)} \sim A^{(2)} \), then also \( A^{(1)} \) decays exponentially.

We now state the following analog of Lemma 3.2.

**Lemma 3.4.** The bi-infinite or semi-infinite block matrices \( A^{(1)} \) and \( A^{(2)} \) of order \( k \) are equivalent if and only if the bi-infinite or semi-infinite matrices \( B^{(1)} \) and \( B^{(2)} \) defined as in (3.1) are equivalent.

**Proof.** We first observe that there exist constants \( c_1, c_2 > 0 \) such that (3.2) holds for any matrix \( C \) of order \( k \), no matter the choice of the matrix norm for the \( k \times k \) matrices. Hence if \( A^{(1)} \) and \( A^{(2)} \) are bi-infinite or semi-infinite block matrices of order \( k \) and \( c > 0 \) and \( \lambda \in (0, 1) \) are such that \( \|A^{(1)}_{ij} - A^{(2)}_{ij}\| \leq c\lambda^{i+j} \) for \( i, j \in J \), then for \( p, q \geq 0 \)

\[
B^{(1)}_{p,q} - B^{(2)}_{p,q} \leq \frac{1}{c_1} \left\| A^{(1)}_{\{x\},\{y\}} - A^{(2)}_{\{x\},\{y\}} \right\| \leq \frac{c}{c_1} \lambda |x-y| \leq \frac{c}{c_1} \lambda \sqrt{\lambda} \left( \sqrt{\lambda} \right)^{|p+q|},
\]
implying that $B^{(1)} \sim B^{(2)}$. Conversely, let $d > 0$ and $\mu \in (0, 1)$ be such that $|B_{p,q}| \leq d\mu^{p-q}$ for $p, q \in J$. Then for $i, j \geq 0$

$$
\|A^{(1)}_{ij} - A^{(2)}_{ij}\| \leq c_2 \sum_{p=k_i}^{k_i+k-1} \sum_{q=k_j}^{k_j+k-1} |B^{(1)}_{p,q} - B^{(2)}_{p,q}| \leq c_2d \sum_{p=k_i}^{k_i+k-1} \sum_{q=k_j}^{k_j+k-1} \mu^{p+q}
$$

$$
\leq c_2d k^2(\mu^k)^{i+j},
$$

which implies $A^{(1)} \sim A^{(2)}$. ■

Lemmas 3.2 and 3.4 of the present paper and Lemmas 3.3, 3.4 and 3.6 of [11] imply the following Lemmas 3.5, 3.6 and 3.8. We therefore omit their proofs. The proof of Lemma 3.7 below is given separately, mimicking the corresponding proof in [11].

**Lemma 3.5.** If $A, B, L, M$ are bi-infinite or semi-infinite block matrices of order $k$ which decay exponentially, and $A \sim L$ and $B \sim M$, then $AB \sim LM$.

**Lemma 3.6.** Suppose that $A$ and $B$ are bi-infinite or semi-infinite block matrices of order $k$ which decay exponentially and have inverses $A^{-1}$ and $B^{-1}$ which decay exponentially. If $A \sim B$, then $A^{-1} \sim B^{-1}$.

**Lemma 3.7.** Let $A$ and $B$ be bi-infinite or semi-infinite block lower triangular matrices of order $k$ which decay exponentially. Suppose $A$ has a block lower triangular bounded inverse $A^{-1}$ on $\ell_2(B)$ or $\ell_2(B_+)$, while $B$ has an inverse $B^{-1}$ which decays exponentially. If $A \sim B$, then $A^{-1} \sim B^{-1}$.

**Proof.** We first consider the special case $B = I$. Choose $c > 0$ and $\lambda \in (0, 1)$ such that

$$
||(I - A)_{ij}|| \leq c\lambda^{i+j}, \quad i, j \geq 0.
$$

Then for $i \geq j \geq 0$ we have

$$
||(A^{-1} - I)_{ij}|| \leq \sum_{l=0}^{\infty}||(I - A)_{il}[A^{-1}]_{lj}|| \leq \|(I - A)_{il}\|_{\ell_2} \|(A^{-1})_{lj}\|_{\ell_2} \leq \left\{ \sum_{l=0}^{\infty} c^2 \lambda^{2(i+l)} \right\}^{1/2} ||A^{-1}||_2
$$

$$
= c^2 \lambda^{i+1/2} ||A^{-1}||_2 \leq \text{const}.(\sqrt{\lambda})^{i+j}.
$$

Moreover, since $|A^{-1} - I|_{il} = 0$ for $i < l$, we conclude that $A^{-1} \sim I$.

Now take $B$ as in the statement of this lemma. Since $A \sim B$, Lemma 3.5 implies $AB^{-1} \sim I$ and also $(AB^{-1})^{-1} = BA^{-1}$ is block lower triangular
and bounded on $\ell_2(\mathbb{Z}_+)$. Therefore, $BA^{-1} \sim I$. As a result of Lemma 3.5 we conclude that $A^{-1} \sim B^{-1}$. ■

**LEMMA 3.8.** If $A$ and $B$ are bi-infinite block matrices of order $k$ that decay exponentially, then

$$(AB)_+ \sim A_+ B_+.$$  

The next lemma states that $L \sim I$ whenever $A = LL^T$ is the block Cholesky factorization of a positive definite semi-infinite block matrix of order $k$ and $A \sim I$. We note that $A \sim I$, which means the existence of $c_F > 0$ and $\lambda \in (0, 1)$ such that $\|A_{ij} - \delta_{ij} I\|_F \leq c_F \lambda^{i+j}$ for any $i, j \geq 0$ and $\|\cdot\|_F$ the Frobenius norm on the $k \times k$ matrices, implies that

$$\|A - I\|^2_F = \sum_{i,j=0}^{\infty} \|A_{ij} - \delta_{ij} I\|^2_F \leq \left(\frac{c_F}{1 - \lambda^2}\right)^2,$$

so that $A - I$ has finite Frobenius norm.

**LEMMA 3.9.** Let $A$ be a bi-infinite or semi-infinite positive definite real symmetric matrix which is bounded on $\ell_2(\mathbb{Z})$ or $\ell_2(\mathbb{Z}_+)$. If $A \sim I$ and $A = LL^T$, where $L$ is a lower block triangular matrix block Cholesky factor of $A$ of order $k$, then $L \sim I$.

**PROOF.** For any semi-infinite block matrix $M = (M_{ij})_{i,j=0}^{\infty}$ of order $k$ and $n \in \mathbb{N}$, let $M_n$ denote the matrix $(M_{ij})_{i,j=0}^{n}$ of order $nk$. Further, let $\tilde{M}_n$ denote the semi-infinite block matrix $(m_{ij})_{i,j=0}^{\infty}$ of order $k$ defined by

$$m_{ij} = \begin{cases} M_{ij}, & 0 \leq i, j \leq n - 1, \\ \delta_{ij} I_k, & \text{otherwise}, \end{cases}$$

where $I_k$ is the identity matrix of order $k$. For $m \geq n$, put $M^m_n = (\tilde{M}_n)_m$.

Since $A$ is positive definite, we can write it in the form $A = LL^T$, where $L$ is its lower block triangular block Cholesky factor of order $k$. Then for all $n$, $A_n = L_n L_n^T$ and $\tilde{A}_n = \tilde{L}_n \tilde{L}_n^T$, while for all $m \geq n$, $A^m_n = L^m_n (L^m_n)^T$.

For each $n$, let $A = \tilde{A}_n + E_n$, where both $\tilde{A}_n$ and $E_n$ are semi-infinite block matrices of order $k$. To apply Theorem A.1 we choose $m \geq n$ and note that $A_m = \tilde{A}^m_n + (E_n)_m$ while for $0 \leq i, j \leq n$, $(E_n)_{ij} = 0$. Since $A \sim I$, we can choose $c_F > 0$ and $\lambda \in (0, 1)$ such that for any $n \in \mathbb{N}$, $\|(E_n)_{ij}\|_F \leq c_F \lambda^{i+j}$ whenever $i \geq n$ or $j \geq n$. Therefore,

$$\|E_n\|^2_F \leq \sum_{i,j=0}^{\infty} c_F^2 \lambda^{2(i+j)} - \sum_{i,j=0}^{n-1} c_F^2 \lambda^{2(i+j)} = c_F^2 \lambda^{2n} \frac{(2 - \lambda^{2n})}{(1 - \lambda^2)^2} \leq K^2 \lambda^{2n},$$

where $K = (c_F \sqrt{2}/(1 - \lambda^2)) > 0$ does not depend on $n$. This implies that

$$\|E_n\|_F \leq K \lambda^n,$$  

(3.3)
while for \( m \geq n \)
\[
\|(E_n)_m\|_F \leq \|E_n\|_F \leq K\lambda^n. \tag{3.4}
\]

Since \( L_m \) is the block Cholesky factor of the matrix \( A_m \),
\[
A_n^m + (E_n)_m = A_m = L_m L_m^T.
\]

Now, using \((3.3)\) we obtain the inequality \( \|A_n^m\| \leq \|\tilde{A}_n\| \leq \|A\| + \|E_n\| \leq \|A\| + \|E_n\|_F \leq \|A\| + K\lambda^n \). Thus for all \( x \in \ell_2(\mathbb{Z}_+) \) we have
\[
\|\tilde{A}_n x\| \geq \left( \|A^{-1}\|^{-1} - K\lambda^n \right) \|x\|, \quad x \in \ell_2(\mathbb{Z}_+).
\]

Hence for any \( m \geq n \) and \( x \in \mathbb{R}^{m+1} \), \( \|A_n^m x\| \geq (\|A^{-1}\|^{-1} - K\lambda^n) \|x\| \). Choosing \( n \in \mathbb{N} \) large enough such that \( \|A^{-1}\|^{-1} - K\lambda^n > 0 \), this gives
\[
\|(A_n^m)^{-1}\| \leq \left( \|A^{-1}\|^{-1} - K\lambda^n \right)^{-1}.
\]

Therefore, letting \( L_m = L_n^m + G_n^m \), by Theorem \( A.1 \), \( L_m = L_n^m + G_n^m \), where
\[
\|G_n^m\|_F \leq \left\| L_n^m \right\| \left\| (E_n)_m \right\|_F \left( \|A_n^m\|^{-1} \right) \left( \|2 - \|(E_n)_m\|_F \|A_n^m\|^{-1}\| \right)
\]
\[
(1 - \|(E_n)_m\|_F \|A_n^m\|^{-1}\|^2)
\]
\[
\leq \frac{2K\|A_n^m\|^{1/2}\lambda^n/(\|A^{-1}\|^{-1} - K\lambda^n)}{\left(1 - \|A^{-1}\|^{-1} - K\lambda^n\right)^2}, \tag{3.5}
\]

where \( \|A_n^m\| \leq \max(1, \|A\|) \). Hence from \((3.4)\) and \((3.5)\) we can choose \( c > 0 \) such that
\[
\|L_m - L_n^m\|_F = \|G_n^m\|_F \leq c\|(E_n)_m\|_F \leq cK\lambda^n,
\]
for all \( n \) large enough and \( m \geq n \). So for \( i \geq n \) or \( j \geq n \), it follows that \( \|L_{ij} - \delta_{ij} I\|_F \leq cK\lambda^n \). In particular, for large enough \( i \) or \( j \), we choose \( n = \max(i, j) \) to obtain
\[
\|L_{ij} - \delta_{ij} I\|_F \leq cK\lambda^n \leq cK(\sqrt{\lambda})^{i+j}.
\]
Thus \( L \sim I \). ■

We now prove

THEOREM 3.10. Suppose that \( A \) and \( B \) are two bi-infinite or semi-infinite positive definite real symmetric block matrices of order \( k \) that decay exponentially, with \( A \sim B \). If \( A = LL^T \) and \( B = MM^T \) are the block Cholesky factorizations of \( A \) and \( B \), then \( L \) and \( L^{-1} \) are lower triangular semi-infinite block matrices of order \( k \) that decay exponentially. Moreover, \( L \sim M \) and \( L^{-1} \sim M^{-1} \).
PROOF. Since $B \sim LL^T$, Lemma 3.5 implies
\[ L^{-1}B(L^{-1})^T \sim L^{-1}LL^T(L^T)^{-1} = I. \]
Recalling Lemma 3.1, we see that whenever $x \in \ell_2(\mathbb{Z}_+)$ also $L^{-1}x \in \ell_2(\mathbb{Z}_+)$. Furthermore, $L^{-1}B(L^{-1})^T$ is positive definite. Since $L$ decays exponentially so does $B = LL^T$. Again, using the exponential decay of $L^{-1}$ we conclude that $L^{-1}B(L^{-1})^T$ decays exponentially and hence is bounded on $\ell_2(\mathbb{Z}_+)$. We now apply Lemma 3.9 to the matrix $L^{-1}B(L^{-1})^T$ and conclude that its block Cholesky factor $N$ of order $k$ defined by the equation
\[ L^{-1}B(L^{-1})^T = NN^T \]
satisfies $N \sim I$. Thus
\[ B = L(L^{-1}B(L^{-1})^T)L^T = LNN^TL^T = MM^T, \]
where $M := LN$. Lemma 3.5 shows that $M = LN \sim L$. Moreover, from the positive definiteness of $B$, there is a constant $c > 0$ such that for all $x \in \ell_2(\mathbb{Z}_+)$, $\|M^Tx\| \geq c\|x\|$. It follows that the lower block triangular inverse $M^{-1}$ of $M$ is bounded on $\ell_2(\mathbb{Z})$. We can then apply Lemma 3.7 to conclude that $M^{-1} \sim L^{-1}$. $\blacksquare$

THEOREM 3.11. Suppose that $A$ is a bi-infinite positive definite real symmetric block matrix of order $k$ which decays exponentially and whose block Cholesky factorization is $A = LL^T$. Then we have the block Cholesky factorization $A_+ = MM^T$ where $M$ decays exponentially and satisfies $M \sim L_+$. Moreover, $M^{-1} \sim L_+^{-1}$.

PROOF. Since $A = LL^T$, Lemma 3.8 shows that
\[ A_+ \sim L_+(L^T)_+ = L_+L_+^T, \]
where $A_+$ is positive definite. The result now follows from Theorem 3.10. $\blacksquare$

As a corollary of this theorem, we obtain a result from [1, 2]. We start with the banded symmetric block Toeplitz matrix $G = (G_i,j)_{i,j \in \mathbb{Z}}$ where the entries $G_{i-j}$ are $k \times k$ matrices. Let $\Gamma = (\Gamma_{i,j})_{i,j \in \mathbb{Z}}$ be its corresponding block Cholesky factor such that its symbol $\Gamma(z)$ is an invertible $k \times k$ matrix if $|z| \leq 1$. Then both $\Gamma$ and $\Gamma^{-1}$ decay exponentially.

COROLLARY 3.12. Let $M$ be the block Cholesky factor of $G_+$. Then
\[ M \sim \Gamma_+. \]

Consequently, there are constants $c > 0$ and $\rho \in (0, 1)$ such that for all $j \geq 0$ and $i \geq j$ we have
\[ \|M_{i,i-j} - \Gamma_j\| \leq c\rho^i. \]
This corollary suggests a numerical method to factorize a Laurent matrix polynomial which is positive on the unit circle. It is a generalization of a method due to Bauer [1, 2] to the block matrix case. Given a Laurent matrix polynomial

\[ G(z) = \sum_{j=-m}^{m} z^j G_j \]

positive definite for \(|z| = 1\), whose coefficients are real \(k \times k\) matrices such that \(G_j = G_j^T\), \(j = 1, \ldots, m\), and \(G_0 = G_0^T\) is positive definite, we consider the bi-infinite banded block Toeplitz matrix

\[ G = (G_{i-j})_{i,j \in \mathbb{Z}} \]

with \(G_{ij} = 0\) for \(|i - j| > m\). As \(G\) is positive definite, its block Cholesky factorization \(G = \Gamma \Gamma^T\) corresponds exactly to the factorization

\[ \Gamma(z)\Gamma(z^{-1})^T = G(z), \quad |z| = 1, \]

of \(G(z)\), where \(\Gamma(z) = \sum_{j=0}^{m} z^j \Gamma_j\), \(\det \Gamma(z)\) has all its roots outside the unit disk and the diagonal elements of \(\Gamma_0\) are positive. Further, the semi-infinite compression \(A_+\) of \(A\) and the finite compressions

\[ G_n = (G_{ij})_{i,j=0}^{n}, \quad n = 0, 1, 2, \ldots, \]

of \(G\) are positive definite. As a result, each finite compression \(G_n, n \in \mathbb{Z}_+\), has a unique block Cholesky factorization

\[ L_n L_n^T = G_n, \]

where \(L_n\) is a lower block triangular matrix of order \(k\) with diagonal \(k \times k\) entries that are positive definite \(k \times k\) matrices. As we increase \(n\) to \(n+1\), the matrix \(G_{n+1}\) agrees with \(G_n\) on the first \((n+1)k\) rows and columns. Likewise \(L_{n+1}\) has its first \((n+1)k\) rows and columns equal to those of \(L_n\). Therefore we may consider \(L_n\) as the \(n\)-th finite section of a semi-infinite block matrix \(L\) of order \(k\) which is the unique block Cholesky factorization of \(G_+\), i.e., \(G_+ = LL^T\).

In the scalar case, Bauer proved that as \(n \to \infty\), the elements of \(L_n\) approach those of \(\Gamma\). Furthermore, in [11] the authors proved that the differences among these elements decay exponentially, that is, there are constants \(c > 0\) and \(\rho \in (0, 1)\) such that for all \(i, j \in \mathbb{Z}_+\)

\[ \|L_{ij} - \Gamma_{i-j}\| \leq c\rho^i, \quad (3.6) \]

where one may use any norm on the \(k \times k\) matrices. Corollary 3.12 states that this result also holds in the block case.

Let \(A\) be a semi-infinite block matrix of order \(k\) and let \((B_n)_{n=0}^{\infty}\) be a sequence of finite block matrices, each of order \(k\) and having the form
$B_n = ([B_n]_{ij})_{i,j=0}^{m_n}$ where $m_n \to +\infty$ as $n \to +\infty$. Then $B_n \asymp A$ if there exist constants $N \in \mathbb{N}$, $c > 0$ and $\lambda \in (0, 1)$ such that for $n \geq N$

\[ ||[B_n]_{ij} - A_{ij}|| \leq c\lambda^{i+j}, \quad 0 \leq i, j \leq m_n. \]

Note that $(||B_n||)_{n=0}^\infty$ is bounded if there exists a semi-infinite block matrix $A$ of order $k$ such that $B_n \asymp A$. Indeed, denoting by $A_n = ([A_n]_{ij})_{i,j=0}^{m_n}$ the truncation of $A$ defined by $[A_n]_{ij} = A_{ij}$ $(0 \leq i, j \leq m_n)$, we observe that for each $n \in \mathbb{Z}_+$

\[ ||B_n|| \leq ||A_n|| + ||B_n - A_n||_F \leq ||A|| + c \left( \sum_{i,j=0}^{m_n} \lambda^{2i+2j} \right)^{1/2} \leq ||A|| + \frac{c}{1 - \lambda^2}. \]

**LEMMA 3.13.** For $n \in \mathbb{N}$, let $A_n$ be a positive definite real symmetric $m_n \times m_n$ matrix whose entries are $k \times k$ matrices. Suppose $m_n \to +\infty$ and $\sup ||A_n^{-1}|| < +\infty$. If $A_n \asymp I$ and $L_n$ is the block Cholesky factor of $A_n$, then $L_n \asymp I$ and $L_n^{-1} \asymp I$.

**PROOF.** Let $A_n \asymp I$. Then there exist constants $N \in \mathbb{N}$, $c > 0$ and $\lambda \in (0, 1)$ such that for $n \geq N$, $||[A_n]_{ij} - \delta_{ij} I_k|| \leq c\lambda^{i+j}$ for $0 \leq i, j \leq m_n$. For $n, m \in \mathbb{N}$ with $m \leq m_n$, we define the matrices $A_{n,m} = ([A_{n,m}]_{ij})_{i,j=0}^{m_n}$ and $E_{n,m} = ([E_{n,m}]_{ij})_{i,j=0}^{m_n}$ by

\[ [A_{n,m}]_{ij} = \begin{cases} [A_n]_{ij}, & 0 \leq \max(i, j) \leq m \\ \delta_{ij} I_k, & m + 1 \leq \max(i, j) \leq m_n \end{cases} \]

and $E_{n,m} = A_n - A_{n,m}$. Then $[E_{n,m}]_{ij} = 0$ for $0 \leq \max(i, j) \leq m$, while for $n \geq N$

\[ ||[E_{n,m}]_{ij}|| \leq c\lambda^{i+j}, \quad m + 1 \leq \max(i, j) \leq m_n. \]

For $n \geq N$ and $m_n \geq m$ we easily check that

\[ ||E_{n,m}||_F \leq c \left( \sum_{m+1 \leq \max(i, j) \leq m_n} \lambda^{2i+2j} \right)^{1/2} \leq \frac{c\sqrt{2}}{1 - \lambda^2} \lambda^m. \]  \hfill (3.7)

Put $\Delta = \sup_{n \in \mathbb{N}} ||A_n^{-1}||$. Choosing $M \in \mathbb{Z}_+$ such that $M > \log(2c\Delta\sqrt{2}/(1 - \lambda^2))/\log(\lambda^{-1})$, for $n \geq N$ and $M \leq m \leq m_n$ we obtain

\[ ||E_{n,m}||_F \leq \frac{1}{2} ||A_n^{-1}||^{-1}, \quad n \geq N. \]  \hfill (3.8)

Now put $\mathcal{E} = \sup_{n \in \mathbb{N}} ||A_n||$ and let $L_{n,m}$ stand for the block Cholesky factor of $A_{n,m}$. Then (3.7), (3.8) and (A.1) imply that

\[ \|[L_n]_{ij} - [L_{n,m}]_{ij}\| \leq ||L_n - L_{n,m}||_F \]

\[ \leq 8||A_n||^{1/2} ||E_{n,m}||_F ||A_n^{-1}|| \leq \frac{8c\Delta\sqrt{2}\mathcal{E}}{1 - \lambda^2} \lambda^m \]
for $n \geq N$, $M \leq m \leq m_n$ and $0 \leq i, j \leq m_n$. This in turn implies

$$\| [L_n]_{ij} - \delta_{ij} I_k \| \leq \frac{8c\Delta \sqrt{2\epsilon}}{1 - \lambda^2} \lambda^m,$$

where $n \geq N$, $M \leq m \leq m_n$ and $m + 1 \leq \max(i, j) \leq m_n$. Hence there exist constants $N \in \mathbb{N}, \ c > 0$ and $\lambda \in (0, 1)$ such that for $n \geq N$

$$\| [L_n]_{ij} - \delta_{ij} I_k \| \leq c \lambda^{i+j}, \quad 0 \leq i, j \leq m_n,$$

and therefore $L_n \simeq I$.

To prove that $L_n^{-1} \simeq I$, we first observe that

$$\| A_n^{-1} - A_{n,m}^{-1} \|_F \| A_n \| \leq \| A_n^{-1} \| \| E_{n,m} \|_F \| A_{n,m}^{-1} \| \| A_n \| \leq \Delta^2 \epsilon \| E_{n,m} \|_F \leq \frac{c \Delta^2 \epsilon \sqrt{2}}{1 - \lambda^2} \lambda^m,$$

where $n \geq N$ and $m_n \geq m$. Choosing $M > \log(2c\Delta^3 \epsilon \sqrt{2}/(1 - \lambda^2))/\log(\lambda^{-1})$, for $n \geq N$ and $M \leq m \leq m_n$ we obtain

$$\| A_n^{-1} - A_{n,m}^{-1} \|_F < \frac{1}{2} \| A_n \|, \quad n \geq N.$$

Using this estimate we now continue the proof as above and conclude that $L_n^{-1} \simeq I$. ■

**THEOREM 3.14.** For $n \in \mathbb{N}$, let $\hat{A}_n$ be a positive definite real symmetric $m_n \times m_n$ matrix whose entries are $k \times k$ matrices. Suppose $m_n \to +\infty$ and $\sup \| \hat{A}_n^{-1} \| < +\infty$. Let $A$ be a positive definite real symmetric block matrix of order $k$ such that $\hat{A}_n \simeq A$. If $\hat{L}_n$ and $L$ are the block Cholesky factors of $\hat{A}_n$ and $A$, respectively, and $L$ and $L^{-1}$ are exponentially decaying, then $\hat{L}_n \simeq L$ and $\hat{L}_n^{-1} \simeq L^{-1}$.

**PROOF.** Let $A_n$ be the truncation of $A$ defined by $[A_n]_{ij} = A_{ij}$ for $0 \leq i, j \leq m_n$. Then there exist constants $N \in \mathbb{N}, \ c, d > 0$ and $\lambda \in (0, 1)$ such that

$$\| L_{ij} \| \leq c \lambda^{i-j}, \quad \| [L^{-1}]_{ij} \| \leq c \lambda^{i-j}, \quad \| [\hat{A}_n]_{ij} - A_{ij} \| \leq d \lambda^{i+j},$$

where $0 \leq i, j \leq m_n$ in the last estimate. Letting $L_n$ be the block Cholesky factor of $A_n$, we consider the matrices

$$C_n = L_n^{-1} \hat{A}_n (L_n^{-1})^T.$$
We now easily estimate
\[
\| [C_n]_{ij} - \delta_{ij} I_k \| \leq \sum_{r=0}^{i} \sum_{s=0}^{j} \| [L_n^{-1}]_{ir} \| \| [\hat{A}_n]_{rs} - [A_n]_{rs} \| \| [L_n^{-1}]_{js} \|
\]
\[
\leq c^2 d \sum_{r=0}^{i} \sum_{s=0}^{j} \lambda_i^{-r} \lambda_j^{r+s} \lambda_i^{-s} = c^2 d (i+1)(j+1) \lambda^{i+j} \leq c^2 d e(\lambda \mu^{-1})^2 \mu^{i+j},
\]
where \( \mu \in (\lambda, 1) \), \( e(t) = \sup_{n \in \mathbb{N}} (n+1)t^n \) for \( t \in (0, 1) \) and \( 0 \leq i, j \leq m_n \). Hence \( C_n \simeq I \).

We now apply Lemma 3.13 to \( (C_n)_{n=1}^{\infty} \), where we observe that (1) the sequence \( (\|C_n^{-1}\|)_{n=1}^{\infty} \) is bounded, (2) \( C_n \simeq I \), and (3) \( L_n^{-1} \hat{L}_n \) is the block Cholesky factor of \( C_n \). Indeed, the boundedness of \( (\|C_n^{-1}\|)_{n=1}^{\infty} \) follows from the estimate
\[
\|C_n^{-1}\| \leq \|L_n\|^2 \|\hat{A}_n^{-1}\| \leq \left( \frac{c}{\lambda^2} \right)^2 \|\hat{A}_n^{-1}\|.
\]
As a result, we obtain \( L_n^{-1} \hat{L}_n \simeq I \) and \( \hat{L}_n^{-1} L_n \simeq I \).

Finally, note that for \( 0 \leq j \leq i \leq m_n \),
\[
\| [L_n]_{ij} \| \leq c \lambda^{i-j} \leq c \lambda^i \leq c.
\]
Further, since \( L_n^{-1} \hat{L}_n \simeq I \) and \( \hat{L}_n^{-1} L_n \simeq I \), there exist constants \( \hat{c}, \hat{N} \in \mathbb{N} \) and \( \nu \in (0, 1) \) such that for \( n \geq \hat{N} \) and \( 0 \leq i, j \leq m_n \),
\[
\| [L_n^{-1} \hat{L}_n]_{ij} - \delta_{ij} I_k \| \leq \hat{c} \nu^{i+j}, \quad \| [\hat{L}_n^{-1} L_n]_{ij} - \delta_{ij} I_k \| \leq \hat{c} \nu^{i+j}.
\]
Hence we have
\[
\| [\hat{L}_n]_{ij} \| \leq \| [L_n]_{ij} \| \left( 1 + \| [L_n^{-1} \hat{L}_n]_{ij} - \delta_{ij} I_k \| \right) \leq c (1 + \hat{c} \nu^{i+j}) \leq c (1 + \hat{c}).
\]
We now get
\[
\| [\hat{L}_n]_{ij} - [L_n]_{ij} \| \leq \| [L_n]_{ij} \| \| [L_n^{-1} \hat{L}_n]_{ij} - \delta_{ij} I_k \| \leq c \hat{c} \nu^{i+j};
\]
\[
\| [\hat{L}_n^{-1}]_{ij} - [L_n^{-1}]_{ij} \| \leq \| [L_n^{-1}]_{ij} \| \| [\hat{L}_n^{-1} L_n]_{ij} - \delta_{ij} I_k \| \leq c \hat{c} \nu^{i+j},
\]
implicating that \( \hat{L}_n \simeq L \) and \( \hat{L}_n^{-1} \simeq L^{-1} \).
§4. Application to the Gram-Schmidt Orthonormalization Process

The properties of the block Cholesky factorization of bi-infinite and semi-infinite positive definite real symmetric stated in Section 3 are a generalization of analogous results for scalar matrices obtained in [12], where they were applied to studying the asymptotic behavior of shifts of an exponentially decaying function. In this section we extend these results to a vector of exponentially decaying functions.

Let \( \varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_\ell(x))^T \), \( x \in \mathbb{R} \), be a vector of real functions defined on \( \mathbb{R} \). We say that \( \varphi \) is an exponentially decaying vector function if there exist constants \( \kappa > 0 \) and \( \lambda \in (0, 1) \) such that

\[
\|\varphi(x)\|_\infty := \max_{i=1, \ldots, \ell} |\varphi_i(x)| \leq \kappa \lambda^{|x|}, \quad x \in \mathbb{R}.
\]

(4.1)

Now consider the integer translates of \( \varphi \), that is the function vectors

\[
\varphi_i := \varphi(\cdot - i), \quad i \in \mathbb{Z},
\]

where \( \varphi_0 := \varphi \). Furthermore, for \( r, s \in \mathbb{Z} \), let the symbol \( < \varphi_r, \varphi_s >_{[a,b]} \) denote the \( \ell \times \ell \) matrix

\[
\left((\varphi_i+r, \varphi_j+s)_{[a,b]}\right)_{i,j=1,\ldots,\ell}
\]

where \((\cdot, \cdot)_{[a,b]}\) is the usual scalar product in \( L^2[a,b] \). According to this definition, \( \varphi_r \) is orthonormal to \( \varphi_s \) on the interval \([a,b]\) if and only if \( < \varphi_r, \varphi_s >_{[a,b]} = I_\ell \delta_{rs} \).

For any such function vector \( \varphi \), the bi-infinite block matrix sequence

\[
T_h := < \varphi_0, \varphi_h >_{\mathbb{R}}, \quad h \in \mathbb{Z},
\]

(4.2)

is even in the sense that \( T_h \) and \( T_{-h} \) are each other's transposes, \( h \in \mathbb{Z} \), and is exponentially decaying since

\[
\|T_h\|_\infty := \max_{i=1, \ldots, \ell} \sum_{j=1}^\ell |[T_h]_{ij}| \leq \kappa^2 \ell \lambda^{|h|} \left(|h| + \frac{1}{\log(\lambda^{-1})}\right).
\]

Hence, by Lemma 3.1, the bi-infinite block Toeplitz matrix of order \( \ell \)

\[
T = (T_{rs})_{r,s \in \mathbb{Z}} = (T_{r-s})_{r,s \in \mathbb{Z}},
\]

is a bounded operator on \( \ell_2(\mathbb{Z}) \).

Now consider the semi-infinite block Gram matrix

\[
G = G_{rs} := < \varphi_r, \varphi_s >_{\mathbb{R}+}, \quad r, s \in \mathbb{Z}_+,
\]

(4.3)

generated by \( \varphi \).
LEMMA 4.1. If $\varphi$ is exponentially decaying, then $G \sim T_+$.

PROOF. By (4.2)-(4.3), for all $r, s \in \mathbb{Z}_+$

$$[T_{rs}]_{ij} - [G_{rs}]_{ij} = \int_{\mathbb{R}_-} \varphi_i(x - r)\varphi_j(x - s)dx, \quad i, j = 1, \ldots, \ell,$$

and, by the inequality (4.1),

$$\|T_{rs} - G_{rs}\|_\infty \leq \kappa^2 \int_{-\infty}^{0} \lambda^{x-r} \lambda^{x-s}dx \leq \kappa^2 \frac{\lambda^{r+s}}{2 \log(\lambda^{-1})}.$$ 

Hence, for some constant $c > 0$

$$\|T_{rs} - G_{rs}\|_\infty \leq c\lambda^{r+s}, \quad r, s \in \mathbb{Z}_+,\]$$

which completes the proof. ■

Now consider the finite block Gram matrices

$$[\widehat{G}_n]_{rs} := <\varphi_r, \varphi_s >_{[0,n]}, \quad r, s = 0, 1, \ldots, n, \quad n \in \mathbb{Z}_+, \quad (4.4)$$

whose block entries for large enough $n, r$ and $s$ are very close to the corresponding entries of $G$, as specified by the following lemma.

LEMMA 4.2. Let $\rho \in (0, 1)$ and consider the truncations $G_n = (G_{ij})^{n}_{i,j=0}$ of $G$. Then there exist $C > 0$ and $\mu \in (0, 1)$ such that

$$\|G_{rs} - \widehat{G}_n\|_{rs} \leq C\mu^{r+s}, \quad 0 \leq r, s \leq [\rho n]. \quad (4.5)$$

Moreover, the sequence of inverses to the matrices $([\widehat{G}_n]_{rs})^{[\rho n]}_{r,s=0}$ is bounded.

PROOF. By (4.1) and (4.3)-(4.4), for $0 \leq \max(r, s) \leq \rho n$ and $i, j = 1, \ldots, \ell$

$$\|\widehat{G}_n - G_n\|_{rs} \leq \kappa^2 \int_{n}^{\infty} \lambda^{x-r} + x-s dx$$

$$= \kappa^2 \frac{\lambda^{2n-r-s}}{2 \log(\lambda^{-1})} = \kappa^2 \frac{\mu^{r+s}}{2 \log(\lambda^{-1})}$$

where $\mu = \lambda^{(1-\rho)/\rho}$. Hence, letting $\| \cdot \|_F$ stand for the Frobenius norm, we have for $0 \leq r, s \leq [\rho n]$

$$\|\widehat{G}_n - G_n\|_{rs} \leq \|\widehat{G}_n - G_n\|_F \leq \ell\kappa^2 \frac{\lambda^{2n-r-s}}{2 \log(\lambda^{-1})} \leq \ell\kappa^2 \frac{\mu^{r+s}}{2 \log(\lambda^{-1})},$$

which proves the first part. The second part follows from the identity

$$\lim_{n \to +\infty} \|([\widehat{G}_n]_{rs} - [G_n]_{rs})^{[\rho n]}_{r,s=0} \| = 0,$$
in combination with the bounded invertibility of $G$ on $\ell_2(\mathbb{Z}_+)$.

Indeed, putting $[\Phi_n]_{rs} = \lambda^{2n-r-s} (0 \leq r, s \leq [\rho n])$, we have the inequality

$$\left\| [\hat{G}_n]_{rs} - [G_n]_{rs} \right\| \leq \frac{\ell \kappa^2}{2 \log(\lambda^{-1})} [\Phi_n]_{rs},$$

where $0 \leq r, s \leq [\rho n]$.

Indeed, according to the Rayleigh-Ritz principle, $\| \hat{G}_n - G_n \|$ does not exceed the spectral norm of the matrix $(\ell \kappa^2/2 \log(\lambda^{-1})) \Phi_n$, which equals its only nonzero eigenvalue; this follows from the fact that $\Phi_n$ has rank one and has only positive elements. From the identity

$$\sum_{s=0}^{[\rho n]} [\Phi_n]_{rs} \lambda^{n-s} = \lambda^{n-r} \sum_{s=0}^{[\rho n]} \lambda^{2n-2s}, \quad 0 \leq r \leq [\rho n],$$

it follows that

$$\left\| \hat{G}_n - G_n \right\| \leq \frac{\ell \kappa^2}{2 \log(\lambda^{-1})} \sum_{s=0}^{[\rho n]} \lambda^{2n-2s} \leq \frac{\ell \kappa^2}{2 \log(\lambda^{-1})} \lambda^{2[(1-\rho)n]+2} \frac{1 - \lambda^{2[\rho n]+2}}{1 - \lambda^2},$$

which vanishes as $n \to +\infty$, as claimed. \[ \blacksquare \]

The first part of Lemma 4.2 can in fact be restated in the form

$$\left( [\hat{G}_n]_{rs} \right)_{r,s=0}^{[\rho n]} \simeq G$$

where $\simeq$ is the relation introduced above Lemma 3.13.

Now suppose that the matrices $T$ and $G$ are positive definite on $\ell_2(\mathbb{Z})$ and $\ell_2(\mathbb{Z}_+)$ respectively. As specified in Theorem 3.11, under this hypothesis there exist the block Cholesky factorizations

$$T = LL^T \quad \text{and} \quad G = MM^T$$

of $T$ and $G$ where $L$ and $L^{-1}$ are bi-infinite lower triangular block Toeplitz matrices with blocks of order $\ell$, $M$ and $M^{-1}$ are semi-infinite lower triangular block matrices, and $L$, $L^{-1}$, $M$ and $M^{-1}$ decay exponentially.

Consider the block Cholesky factorization

$$\hat{G}_n = \hat{M}_n \hat{M}_n^T, \quad n \in \mathbb{Z}_+.$$

It is straightforward to prove that for all $i = 0, 1, \ldots, n$ and $x \in \mathbb{R}$ the function vectors $\psi_i^n$ defined by

$$\psi_i^n(x) = \sum_{j=0}^{i} \left[ \hat{M}_n^{-1} \right]_{ij} \varphi_j(x) \quad (4.6)$$
are the ones generated by the block Gram-Schmidt orthonormalization process applied to \( \{ \varphi_j \}_{j=0,1,\ldots,n} \) on \([0,n]\).

Under the above hypothesis on the Toeplitz matrix \( T \), the function vector

\[
\psi(x) := \sum_{j \in \mathbb{Z}_+} [L^{-1}]_j \varphi_{-j}(x), \quad x \in \mathbb{R},
\]

(4.7)
decays exponentially. Furthermore, its integer translates are orthonormal on \( \mathbb{R} \), that is

\[
< \psi_r, \psi_s >_{\mathbb{R}} = I_\ell \delta_{rs}, \quad r, s \in \mathbb{Z},
\]

with \( \psi_h := \psi(\cdot - h), \ h \in \mathbb{Z} \).

The main result of this section, which is a generalization of Theorem 2.1 of [12], is that, under the conditions claimed in Theorem 4.4, the function vector \( \psi \) defined above supplies the limiting profile of the block Gram-Schmidt orthonormalization process applied to the interval \([0,n]\), with large enough \( n \).

Using formulae (4.1) and (4.6)-(4.7) we easily obtain that, for all \( i = 0,1,\ldots,n \) and \( x \in \mathbb{R} \),

\[
\| \psi_i^h(x) - \psi_i(x) \| \leq \kappa \sum_{j=0}^{i} \| [\mathcal{M}_n^{-1}]_{ij} - [L^{-1}]_{ij} \| + \kappa \sum_{j=-\infty}^{-1} \| [L^{-1}]_{ij} \|.
\]

Since \([L^{-1}]_{ij} = [L^{-1}]_{i-j}\) and the sequence \([L^{-1}]_h, \ h \in \mathbb{Z}_+\), decays exponentially as \( h \to \infty \), there are constants \( c > 0 \) and \( \alpha \in (0,1) \) such that

\[
\| [L^{-1}]_{ij} \| \leq c \alpha^{|i-j|}, \quad i, j = 0,1,\ldots,n.
\]

(4.8)

This inequality implies that, for \( x \in \mathbb{R} \),

\[
\| \psi_i^h(x) - \psi_i(x) \| \leq \kappa \sum_{j=0}^{i} \| [\mathcal{M}_n^{-1}]_{ij} - [L^{-1}]_{ij} \| + \kappa \frac{\alpha^i}{1 - \alpha}, \quad i = 0,1,\ldots,n.
\]

(4.9)

**THEOREM 4.3.** Suppose \( \varphi \) is a real function vector on \( \mathbb{R} \) which satisfies the inequality (4.1) and has the property that the corresponding block Toeplitz matrix \( T \) and the Gram matrix \( G \) are positive definite. Then for every constant \( \rho \in (0,1) \) there are constants \( C > 0, \mu \in (0,1) \) and \( N \in \mathbb{Z}_+ \) such that for all \( i,n \in \mathbb{Z}_+ \), with \( 0 \leq i \leq \rho n \) and \( n \geq N \) we have

\[
\| \psi_i^n(x) - \psi_i(x) \| \leq C \mu^i, \quad x \in \mathbb{R},
\]

where \( \psi \) is defined by (4.7).

**PROOF.** Let \( \tilde{G}_n \) and \( \tilde{M}_n \) stand for the submatrices of \( \tilde{G}_n \) and \( \tilde{M}_n \) of order \([\rho n] + 1 \) (with \( \ell \times \ell \) matrices as their entries) composed of the \((i,j)\)-elements with \( 0 \leq i,j \leq [\rho n] \). Since \( \tilde{M}_n \) is lower triangular, \( \tilde{G}_n = \tilde{M}_n \tilde{M}_n^T \) is the block Cholesky factorization of \( \tilde{G}_n \). It suffices to prove that \( \tilde{M}_n^{-1} \simeq L^{-1}, \)
because this implies the existence of constants $N \in \mathbb{N}$, $c > 0$ and $\lambda \in (0, 1)$ such that
\[
\left\| (\tilde{M}_n^{-1})_{ij} - [L^{-1}]_{ij} \right\| \leq c \lambda^{i+j}, \quad 0 \leq i, j \leq [\rho n].
\]
With the help of (4.9), this would in turn imply
\[
\| \psi^*_i(x) - \psi_i(x) \| \leq \kappa \left( \frac{c}{1 - \lambda} \lambda^i + \frac{1}{1 - \alpha} \alpha^i \right) \leq C \mu^i, \quad i = 0, 1, \ldots, [\rho n],
\]
where $\mu = \max(\lambda, \alpha)$ and $C = \kappa \{ (c/(1 - \lambda)) + (1/(1 - \alpha)) \}$.

Now recall that $\tilde{G}_n \simeq G$, according to the first part of Lemma 4.2. Thus to apply Theorem 3.14, in view of the exponential decay of $L$ and $L^{-1}$, it is sufficient to prove that $\sup_{n \in \mathbb{N}} \| \tilde{G}_n^{-1} \| < +\infty$. But this follows from the second part of Lemma 4.2. ■

§5. An Example

We shall now give a practical illustration of the procedure for the identification of the limiting profile proposed above. We feel that all the results obtained in [11] on this topic can be extended to vectors of compactly supported functions. Therefore a suitable choice for this illustration is to consider a vector of two B-splines.

Let $B_1$ and $B_2$ the following two B-splines with support $[0, 3]$ and degrees 3 and 4, respectively:

\[
B_1(x) = -\frac{11}{36} (-x)_+^3 i - \frac{1}{2} (-x)_+^2 + \frac{1}{2} (1 - x)_+^3
\]
\[
- \frac{1}{4} (2 - x)_+^3 + \frac{1}{18} (3 - x)_+^3,
\]
\[
B_2(x) = \frac{1}{12} (3 - x)_+^4 - 2 (2 - x)_+^3 + \frac{3}{4} (2 - x)_+^4
\]
\[
- \frac{3}{4} (1 - x)_+^4 - 2 (1 - x)_+^3 - \frac{1}{12} (-x)_+^4.
\]

We now take $\varphi(x) = (B_1(x), B_2(x))^T$, $x \in \mathbb{R}$, and consider its integer translates $\varphi_i(x) := \varphi(x - i)$, $i \in \mathbb{Z}$.

Let $T$ and $G$ be the Gram matrices (4.2) and (4.3) generated by the integer and the positive integer translates of $\varphi$, respectively. It is immediate that both of them are 5-diagonal block matrices with blocks of order two.

The symbol of the bi-infinite matrix $T$ is
\[
\Sigma(z) = \frac{1}{\alpha} \left( T_2 z^2 + T_1 z + T_0 + T_{-1} z^{-1} + T_{-2} z^{-2} \right),
\]
where $\alpha = 362880$, $T_{-2}$ and $T_{-1}$ are the transposes of $T_2$ and $T_1$, respectively, and

$$
T_0 = \begin{bmatrix} 13176 & 10179 \\ 10179 & 11304 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 4634 & 6573 \\ 1275 & 1688 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 124 & 111 \\ 6 & 4 \end{bmatrix}.
$$

As the computation of the zeros of $\text{det}(z^2 \Sigma(z))$ proves, the symbol $\Sigma(z)$ is positive for $|z| = 1$ and has four eigenvalues inside and four outside the unit circle. Hence the Toeplitz matrix $T$ is positive definite. Furthermore, $G$ is positive definite too, since in this case $G = T_+$, so that the hypotheses of Theorem 4.3 are satisfied.

As in (4.6), let $\psi^n_i(x)$, $i = 0, 1, \ldots, n$, be the function vectors generated by the Gram-Schmidt orthonormalization process applied to $\varphi_j(x)$, $j = 0, 1, \ldots, n$, on $[0, n]$. Figure 1 illustrates the two components of $\psi^{20}_i(x)$ for some values of $i$.

Let $\psi(x)$ be the limiting profile vector (4.7) arising from the asymptotic orthonormalization process applied to $\varphi_i(x)$, $i \in \mathbb{Z}$. A graph of the two components of $\psi(x)$ is sketched in Figure 2. We shall illustrate the method, implicit in Theorem 2.1, for obtaining the coefficients $[L^{-1}]_j$ appearing in (4.7), in detail.

Since $T$ is positive definite, the symbol $\Sigma(z)$ has a factorization of the form (2.7), where $\mathcal{M}_1 = L_1$, $\mathcal{M}_2 = L_2$ and $\mathcal{D}$ is positive definite. The matrices $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{D}$ can be computed by the identities (2.18) without the need to consider Jordan chains, as the eigenvalues are all simple.

By formulae (2.19) we obtain

$$L(z) = (I_2 - z\mathcal{M}_1^T - z^2\mathcal{M}_2^T)\mathcal{D}^{1/2}$$

which can be factorized in the form

$$L(z) = (I_2 - zR_1)(I_2 - zR_2)\mathcal{D}^{1/2}$$
by the method used in the proof of Theorem 2.1.

As the eigenvalues $\mu_i$, $i = 1, 2, 3, 4$, of $R_1$ and $R_2$ are the eigenvalues of $\Sigma(z)$ inside the unit circle, we have $\|R_1\| < 1$ and $\|R_2\| < 1$. Hence, expanding $(I_2 - zR_1)^{-1}$ and $(I_2 - zR_2)^{-1}$ into two power series and setting

$$[L(z)]^{-1} = D^{-1/2}(I_2 - zR_2)^{-1}(I_2 - zR_1)^{-1} := \sum_{j \in \mathbb{Z}_+} [L^{-1}]_j z^j,$$

we obtain

$$[L^{-1}]_j = D^{-1/2} \sum_{h=0}^j R_2^h R_1^{j-h}, \quad j \in \mathbb{Z}_+.$$

§6. Conclusions

As remarked in the introduction, the limiting profile arising from the orthonormalization of the positive integer translates of a B-spline with equally spaced knots has been identified in [11]. The analogous problem for an exponentially decaying function has been solved in [12]. In a future paper we intend to generalize these results to vectors whose elements are either B-splines or exponentially decaying functions.

Appendix A: Block Cholesky Factorization of Perturbed Matrices

By $\mathcal{F}$ we denote the class of bi-infinite and semi-infinite (block) matrices with finite Frobenius norm and by $\|B\|_F$ the Frobenius norm of $B$. Here we note that the class $\mathcal{F}$ and the value of the norm in $\mathcal{F}$ are not affected by partitioning the matrix $B$ into blocks that are $k \times k$ matrices.

In this appendix we prove the following result, which is closely related to Theorem 1.1 of [16]. This theorem is also valid for finite positive definite real symmetric matrices.
THEOREM A.1. Let $A$ be a bi-infinite or semi-infinite positive definite real symmetric matrix with block Cholesky factorization $A = LL^T$ of order $k$. If $E$ is a real symmetric matrix satisfying $\|E\|_F < \|A^{-1}\|^{-1}$, then the unique block Cholesky factorization $A + E = (L + G)(L + G)^T$ of order $k$ satisfies
\[
\|G\|_F \leq \frac{\|A\|^{1/2}\|E\|_F\|A^{-1}\|(2 - \|E\|_F\|A^{-1}\|))}{(1 - \|E\|_F\|A^{-1}\|)^2}.
\]  

PROOF. Let $P$, $Q$ and $R$ be the projections that transform a bi-infinite or semi-infinite block matrix of order $k$ into its strictly upper block triangular, strictly lower block triangular and block diagonal part, respectively. Then the Frobenius norms of $PB$, $QB$, $(P + R)B$ and $(Q + R)B$ do not exceed $\|B\|_F$ for any $B \in \mathcal{F}$.

Let us first assume that $B \in \mathcal{F}$ and $\|B\|_F < 1$, where $B = I - A$. Following [5], let $X$, $Y$, $Z$ and $W$ be the unique solutions of the equations
\[
X - (P + R)BX = I \quad (A.2)
\]
\[
Y - (Q + R)YB = I \quad (A.3)
\]
\[
Z - PBZ = I \quad (A.4)
\]
\[
W - QBW = I, \quad (A.5)
\]
where their uniqueness is clear from $\|B\|_F < 1$ and $X - I$, $Y - I$, $Z - I$ and $W - I$ belong to $\mathcal{F}$. Then $(I - B)X = I + (P + R)BX - BX = I - QBX$ implies
\[
(I - B)(I + (P + R)BX) = I - QBX. \quad (A.6)
\]

Similarly, $W(I - B) = I + QBW - WB = I - (P + R)WB$ implies
\[
(I + QBW)(I - B) = I - (P + R)WB. \quad (A.7)
\]

Hence, from (A.6) and (A.7) we obtain
\[
(I + QBW)(I - QBX) = (I + QBW)(I - B)[I + (P + R)BX]
\]
\[
= [I - (P + R)WB][I + (P + R)BX]. \quad (A.8)
\]

Since the ranges of the projections $Q$ and $P + R$ have zero intersection, the two sides of (A.8) equal the identity operator. Further, since all factors in (A.8) are compact perturbations of the identity operator, they are boundedly invertible operators on $\ell_2(J)$ where $J = Z$ or $J = Z_+$. Thus $I + (P + R)BX$ and $I - (P + R)WB$ are each others inverses and
\[
I - B = (I - QBX)[I - (P + R)WB]. \quad (A.9)
\]

Similarly, since $Y(I - B) = I - PYB$ and $(I - B)Z = I - (Q + R)BZ$ imply the identities $[I + (Q + R)YB](I - B) = I - PYB$ and $(I - B)(I + PBZ) = I - (Q + R)BZ$, which imply
\[
[I + (Q + R)YB][I - (Q + R)BZ] = (I - PYB)(I + PBZ), \quad (A.10)
\]
and since both sides of (A.10) are equal to the identity matrix, we obtain
\[ I - B = [I - (Q + R)BZ](I - PYB). \tag{A.11} \]

Using (A.9) and (A.11) we get the identity
\[ [I - (P + R)WB](I - PYB)^{-1} = (I - QBX)^{-1}[I - (Q + R)BZ], \]
which can be written as \( I - D \) for some \( D \in \text{Im } R \), because \( (I - PYB)^{-1} = I + PBZ \), \( (I - QBX)^{-1} = I + QWB \), and \( \text{Im } (P + R) \cap \text{Im } (Q + R) = \text{Im } R \). Therefore for some \( D \in \text{Im } R \)
\[ I - B = (I - QBX)(I - D)(I - PYB), \]
and further
\[ I - (P + R)WB = (I - D)(I - PYB), \quad I - (Q + R)BZ = (I - QBX)(I - D). \]

Equating the diagonal parts we obtain
\[ D = RBZ = RWB. \]

As a result, since \( A = I - B \) is a positive definite real symmetric matrix, we write the block Cholesky factorization of \( A \) in the form \( A = LL^T \), where
\[ L = (I - QBX)(I - RBZ)^{1/2}. \]

Iterating (A.2)-(A.5) one now easily proves that
\[ \max\{\|QBX\|_F, \|(I - RBZ)^{1/2} - I\|_F\} \leq \frac{\|B\|_F}{1 - \|B\|_F}, \]
which implies
\[ \|L - I\|_F \leq \frac{2\|B\|_F}{1 - \|B\|_F} + \left( \frac{\|B\|_F}{1 - \|B\|_F} \right)^2 = \frac{\|B\|_F(2 - \|B\|_F)}{(1 - \|B\|_F)^2}. \tag{A.12} \]

Hence, if \( LL^T = I - B \) is the block Cholesky factorization of the semi-infinite matrix \( I - B \) with \( \|B\|_F < 1 \), then (A.12) is valid.

Let us now consider the general case where we do not assume \( B \in \mathcal{F} \). Using the positive definiteness of the real matrix \( A \) and \( A = LL^T \) we readily find the estimate
\[ \|L^{-1}E(L^{-1})^T\|_F^2 = \text{Tr}(L^{-1}E(L^{-1})^T L^{-1} E^T(L^{-1})^T) \]
\[ = \text{Tr}(L^{-1}E A^{-1} E^T(L^{-1})^T) = \text{Tr}((L^{-1})^T L^{-1} E A^{-1} E^T) \]
\[ = \text{Tr}(A^{-1} E A^{-1} E^T) = \text{Tr}(A^{-1/2} E A^{-1} E^T A^{-1/2}) \]
\[ = \|A^{-1/2} E A^{-1/2}\|_F^2 \leq \|A^{-1/2}\|_F^4 \|E\|_F^2 = \|A^{-1}\|_F^2 \|E\|_F^2, \]
which implies
\[ \| L^{-1} E (L^{-1})^T \|_F \leq \| A^{-1} \| \| E \|_F. \]  
(A.13)

Putting \( M = L^{-1} (L + G) \) we now find
\[
\begin{align*}
L^{-1} (A + E) (L^{-1})^T &= I + L^{-1} E (L^{-1})^T \\
L^{-1} (A + E) (L^{-1})^T &= L^{-1} (L + G) (L + G)^T (L^{-1})^T = MM^T,
\end{align*}
\]
where, because of (A.13),
\[ \| M - I \|_F \leq \frac{\| E \|_F \| A^{-1} \| \left( 2 - \| E \|_F \| A^{-1} \| \right)}{(1 - \| E \|_F \| A^{-1} \|)^2}. \]  
(A.14)

Consequently, as a result of (A.12), (A.14), \( G = L (M - I) \) and \( \| L \| = \| A \|^{1/2} \) (which follows from the estimate \( \| L x \|^2 = (L L^T x, x) = \| A^{1/2} x \|^2 \) and the identity \( \| L \| = \| L^T \| \) we get (A.1), which completes the proof.

The proof of Theorem A.1 crucially depends on the boundedness of the projections \( P \) and \( Q \) onto the strictly upper and strictly lower block triangular parts of a semi-infinite matrix in \( \mathcal{F} \). These projections are no longer bounded if \( \mathcal{F} \) is replaced by the Banach algebra of all bounded semi-infinite block Toeplitz matrices. Indeed ([4], Example 4.1), the semi-infinite Toeplitz matrix \( G = (G_{i-j})_{i,j \in \mathbb{Z}} \) given by \( G_0 = 0 \) and \( G_s = 1/s \) for \( s \neq 0 \) is bounded with norm \( \leq \pi \), but the norms of the strictly upper triangular parts of its antisymmetric \( n \times n \) sections have a norm \( \geq (4/5) \log n \). Hence the projections \( P \) and \( Q \) are unbounded on the algebra of bounded bi-infinite Toeplitz matrices. Note that this matrix \( G \) does not belong to the Wiener algebra.

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