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WAVE SCATTERING IN 1-D NONCONSERVATIVE MEDIA

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ABSTRACT

In this review paper, the generalized Schrödinger equation $d^2\psi/dx^2 + k^2\psi = [ikP(x) + Q(x)]\psi$ is considered, where $P(x)$ and $Q(x)$ are real, integrable potentials with finite first moments. The scattering solutions and the bound state solutions are studied, the scattering coefficients and their small-k and large-k asymptotics are analyzed. Unless $P(x) \leq 0$, it is shown that there may be bound states at complex energies, degenerate bound states, and singularities of the transmission coefficient for real k. Some illustrative examples are provided.

1. INTRODUCTION

In this paper we are interested in analyzing the scattering problem for

$$-\psi^{+\prime\prime}(k,x) + [ikP(x) + Q(x)]\,\psi^+(k,x) = k^2\psi^+(k,x), \qquad x \in \mathbf{R}, \tag{1.1}$$

where $\mathbf{R}$ is the real line, the prime denotes the derivative with respect to x, k is a complex parameter, and $P(x)$ and $Q(x)$ are real-valued functions such that $P, Q \in L^1_1(\mathbf{R})$. By $L^1_1(\mathbf{R})$ we denote the Lebesgue-measurable functions $f(x)$ such that $||f||_{1,1}$ is finite, where $||f||_{1,1} = \int_{-\infty}^{\infty} dx\,(1+|x|)\,|f(x)|$. Most of our results are valid under the weaker assumptions $P, Q \in L^1(\mathbf{R})$, and the first moments are needed only when we consider (1.1) near or at $k = 0$. The reason for us to use the superscript plus in (1.1) will be apparent in Section 3: Associated with (1.1) is the related equation (3.1), where the superscript minus appears.

There are several reasons why the analysis of (1.1) is important. In quantum mechanics, (1.1) describes the behavior of a particle of momentum k and energy k^2 interacting with the energy-dependent potential $ikP(x)+Q(x)$. In this case $\psi^+(k,x)$ corresponds to the wavefunction. Moreover, in the frequency domain, (1.1) describes the wave propagation in a one-dimensional medium where $P(x)$ represents energy absorption or generation in the medium and $Q(x)$ is the restoring force density. In this context, the time-domain analog of (1.1) is given by

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}-P(x)\frac{\partial u}{\partial t}=Q(x)\,u, \qquad t,x\in\mathbf{R},$$

where the wave speed is equal to 1. When $P(x)\le 0$, there is net absorption; however, unless otherwise stated we will not put any restriction on the sign of $P(x)$.

The scattering states of (1.1) correspond to its solutions behaving like e^{ikx} or e^{-ikx} as $x\to\pm\infty$ for $k\in\mathbf{R}$. As indicated in Section 2, such solutions can be extended continuously and analytically in k to the upper-half complex plane $\mathbf{C}^+$. If at a certain k-value in $\mathbf{C}^+$ such solutions decay exponentially as $x\to\pm\infty$, we obtain a bound state. A bound state of (1.1) corresponds to a nontrivial solution belonging to $L^2(\mathbf{R})$.

When $P(x)\equiv 0$, (1.1) is reduced to the usual Schrödinger equation

$$-\psi^{[0]\prime\prime}(k,x)+Q(x)\,\psi^{[0]}(k,x)=k^2\psi^{[0]}(k,x), \qquad x\in\mathbf{R}. \tag{1.2}$$

When $Q\in L^1(\mathbf{R})$, the scattering theory for (1.2) is well understood. Let us use $N(P,Q)$ to denote the number of bound states (including multiplicities) of (1.1); hence $N(0,Q)$ denotes the number of bound states of (1.2). When $Q\in L^1(\mathbf{R})$ there may be infinitely many bound states of (1.2) all having negative energies accumulating at zero, but each bound state is simple. When $N(0,Q)$ is finite, we let $k=i\kappa_j$ for $j=1,\cdots,\mathcal{N}$ denote the corresponding momentum values with $0<\kappa_1<\cdots<\kappa_{\mathcal{N}}$. If $Q\in L^1_1(\mathbf{R})$, it is assured that $N(0,Q)$ is finite. Since $P(x)$ is real, unless $P(x)\equiv 0$, (1.1) is a non-self-adjoint equation and there may be complex eigenvalues, i.e. values of k^2 at which there exist solutions belonging to $L^2(\mathbf{R})$. Furthermore, (1.1) may also have eigenvalues that are not simple. Some examples of complex or multiple eigenvalues will be given in Section 8.

This paper is organized as follows. In Section 2 we consider the Jost solutions of (1.1), i.e. certain scattering solutions in terms of which the scattering coefficients are defined. In Section 3 we study the scattering coefficients and some of their properties. Section 4 is about the large-k and small-k asymptotics of the scattering coefficients. In Section 5 the bound states of (1.1) are considered, some estimates are obtained on the number of bound states, and a Levinson theorem is presented relating the number of bound states to the argument of the transmission coefficient. In Section 6 we show that the negative-energy bound states of (1.1) can be analyzed in terms of the eigenvalue branches of the operator $\mathcal{O}_\beta$ defined in (6.1). Section 7 is devoted to the connection between the zeros of the Jost solutions of (1.1) and the bound states. In Section 8 we present various examples illustrating the theory contained in the prior sections. Finally, in Section 9 we conclude with a brief summary of some inverse scattering problems for (1.1).

Since this is a review paper, no proofs are included. For proofs and further details, we refer the reader to [1] and the references therein.

2. SCATTERING SOLUTIONS

Among the scattering solutions of (1.1), we have the Jost solution from the left $f_l^+(k,x)$ and the Jost solution from the right $f_r^+(k,x)$ satisfying the boundary conditions

$$f_l^+(k,x) = e^{ikx} + o(1), \qquad f_l^{+\prime}(k,x) = ike^{ikx} + o(1), \qquad x \to +\infty, \tag{2.1}$$

$$f_r^+(k,x) = e^{-ikx} + o(1), \qquad f_r^{+\prime}(k,x) = -ike^{-ikx} + o(1), \qquad x \to -\infty. \tag{2.2}$$

These solutions satisfy

$$f_l^+(k,x) = \frac{1}{T^+(k)} e^{ikx} + \frac{L^+(k)}{T^+(k)} e^{-ikx} + o(1), \qquad x \to -\infty, \tag{2.3}$$

$$f_r^+(k,x) = \frac{1}{T^+(k)} e^{-ikx} + \frac{R^+(k)}{T^+(k)} e^{ikx} + o(1), \qquad x \to +\infty, \tag{2.4}$$

where $T^+(k)$ is the transmission coefficient, and $R^+(k)$ and $L^+(k)$ are the reflection coefficients from the right and from the left, respectively. Let $[F;G] = FG' - F'G$ denote the Wronskian. Note that the same transmission coefficient appears in both (2.3) and (2.4); this is because $[f_l^+(k,x);f_r^+(k,x)]$ is independent of x, and the values of that Wronskian at $x = \pm\infty$ show that the transmission coefficient from the left is the same as the transmission coefficient from the right.

Define the Faddeev functions from the left $m_l^+(k,x)$ and from the right $m_r^+(k,x)$ as

$$m_l^+(k,x) = e^{-ikx} f_l^+(k,x), \qquad m_r^+(k,x) = e^{ikx} f_r^+(k,x).$$

By $\overline{\mathbf{C}^+}$ we denote $\mathbf{C}^+ \cup \mathbf{R}$. In the next theorem we show that, for each fixed x, the Jost solutions can be analytically extended to $\mathbf{C}^+$.

Theorem 2.1. *Assume $P, Q \in L^1(\mathbf{R})$. Then, for each $x \in \mathbf{R}$, the functions $m_l^+(k,x)$, $m_r^+(k,x)$, $m_l^{+\prime}(k,x)$, and $m_r^{+\prime}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$. Similarly, for each $x \in \mathbf{R}$, the functions $f_l^+(k,x), f_r^+(k,x), f_l^{+\prime}(k,x)$, and $f_r^{+\prime}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$. Moreover, we have*

$$|m_l^+(k,x)| \le Ce^{C/|k|}, \qquad |m_r^+(k,x)| \le Ce^{C/|k|}, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\},$$

$$|m_l^{+\prime}(k,x)| \le C(1+|k|)e^{C/|k|}, \qquad |m_r^{+\prime}(k,x)| \le C(1+|k|)e^{C/|k|}, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\},$$

where C is a constant independent of x and k. If we further assume $Q \in L^1_1(\mathbf{R})$, then the Faddeev functions, the Jost solutions, and their x-derivatives are continuous at $k = 0$ as well. In this case, for $k \in \overline{\mathbf{C}^+}$ we have

$$|m_l^+(k,x)| \le C[1+\max\{0,-x\}], \qquad |m_r^+(k,x)| \le C[1+\max\{0,x\}],$$

$$|m_l^{+\prime}(k,x)| \le C[1+|k|][1+\max\{0,-x\}], \quad |m_r^{+\prime}(k,x)| \le C[1+|k|][1+\max\{0,x\}].$$

Theorem 2.2. *Assume $P, Q \in L^1(\mathbf{R})$. For each $k \in \overline{\mathbf{C}^+}$, the quantities $m_l^+(k,x)$, $m_r^+(k,x)$, $m_l^{+\prime}(k,x)$, and $m_r^{+\prime}(k,x)$ are bounded and continuous, and we have*

$$m_l^+(k,x) = \begin{cases} 1+o(1), & x \to +\infty, \\ \frac{1}{T^+(k)} + o(1), & x \to -\infty, \end{cases}$$

$$m_r^+(k,x) = \begin{cases} \frac{1}{T^+(k)} + o(1), & x \to +\infty, \\ 1+o(1), & x \to -\infty, \end{cases}$$

$$m_l^{+\prime}(k,x) = o(1), \qquad m_r^{+\prime}(k,x) = O(1), \qquad x \to +\infty,$$

$$m_l^{+\prime}(k,x) = O(1), \qquad m_r^{+\prime}(k,x) = o(1), \qquad x \to -\infty.$$

Proposition 2.3. *Assume $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$. Then, $f_l^+(0,x)$ and $f_r^+(0,x)$ are determined by $Q(x)$ alone, and we have*

$$f_l^+(0,x) = f_l^{[0]}(0,x), \qquad f_r^+(0,x) = f_r^{[0]}(0,x). \tag{2.5}$$

Generically $f_l^{[0]}(0,x)$ and $f_r^{[0]}(0,x)$ are linearly independent, but in the exceptional case these two functions are linearly dependent [2–4]. From Proposition 2.3 we see that $Q(x)$ alone determines whether we are in the generic or exceptional case. In the exceptional case, let us define

$$\gamma = \frac{f_l^{[0]}(0,x)}{f_r^{[0]}(0,x)}. \tag{2.6}$$

Then γ is a real, nonzero constant determined by $Q(x)$ alone, and we have $\gamma = f_l^{[0]}(0,-\infty)$ and $\gamma = 1/f_r^{[0]}(0,+\infty)$.

The transformation $k \mapsto -\bar{k}$ in the complex plane is a reflection with respect to the imaginary axis, where the overline denotes complex conjugation. Under this transformation, we have $ik \mapsto \overline{ik}$ and

$$f_l^{\pm}(-\bar{k},x) = \overline{f_l^{\pm}(k,x)}, \qquad f_r^{\pm}(-\bar{k},x) = \overline{f_r^{\pm}(k,x)}, \qquad k \in \overline{\mathbf{C}^+}. \tag{2.7}$$

Hence, for real k, we get

$$f_l^{\pm}(-k,x) = \overline{f_l^{\pm}(k,x)}, \qquad f_r^{\pm}(-k,x) = \overline{f_r^{\pm}(k,x)}, \qquad k \in \mathbf{R}.$$

3. SCATTERING COEFFICIENTS

The transmission coefficient $T^+(k)$ given in (2.3) and (2.4) can also be defined in terms of a Wronskian of the Jost solutions of (1.1). However, this is not true for the reflection coefficients. In order to write the reflection coefficients of (1.1) in terms of Wronskians of the Jost solutions, we also need to consider the differential equation

$$-\psi^{-\prime\prime}(k,x) + [-ikP(x) + Q(x)]\,\psi^-(k,x) = k^2\psi^-(k,x), \qquad x \in \mathbf{R}. \tag{3.1}$$

Notice that (3.1) is obtained from (1.1) by changing the sign of $P(x)$. Let $f_l^-(k,x)$ and $f_r^-(k,x)$ denote the Jost solutions of (3.1) from the left and from the right, respectively, satisfying the boundary conditions (2.1) and (2.2), respectively. As in (2.3) and (2.4), in terms of the spatial asymptotics of these Jost solutions we can define the transmission coefficient $T^-(k)$, the reflection coefficient from the right $R^-(k)$, and the reflection coefficient from the left $L^-(k)$. In terms of the Jost solutions of (1.1) and (3.1), we have

$$[f_l^\pm(k,x);f_r^\pm(k,x)] = -\frac{2ik}{T^\pm(k)}, \qquad k\in\overline{\mathbf{C}^+}, \tag{3.2}$$

$$[f_l^\pm(k,x);f_r^\mp(-k,x)] = \frac{2ikL^\pm(k)}{T^\pm(k)} = -\frac{2ikR^\mp(-k)}{T^\mp(-k)}, \qquad k\in\mathbf{R}, \tag{3.3}$$

$$[f_r^\pm(k,x);f_l^\mp(-k,x)] = -\frac{2ikR^\pm(k)}{T^\pm(k)} = \frac{2ikL^\mp(-k)}{T^\mp(-k)}, \qquad k\in\mathbf{R}. \tag{3.4}$$

The scattering matrices $\mathbf{S}^+(k)$ associated with (1.1) and $\mathbf{S}^-(k)$ associated with (3.1) are defined as

$$\mathbf{S}^\pm(k) = \begin{bmatrix} T^\pm(k) & R^\pm(k) \\ L^\pm(k) & T^\pm(k) \end{bmatrix}.$$

Let $\mathbf{S}^{[0]}(k)$ denote the scattering matrix associated with (1.2):

$$\mathbf{S}^{[0]}(k) = \begin{bmatrix} T^{[0]}(k) & R^{[0]}(k) \\ L^{[0]}(k) & T^{[0]}(k) \end{bmatrix},$$

where $T^{[0]}(k)$ is the transmission coefficient, and $R^{[0]}(k)$ and $L^{[0]}(k)$ are the reflection coefficients from the right and from the left, respectively. When $Q\in L_1^1(\mathbf{R})$, it is known that $\mathbf{S}^{[0]}(k)$ exists and is continuous for $k\in\mathbf{R}$. From (2.7) and (3.2) we obtain

$$\frac{k}{T^\pm(-\overline{k})} = \frac{k}{\overline{T^\pm(k)}}, \qquad k\in\overline{\mathbf{C}^+}.$$

Although the quantities given on the right-hand sides of (3.2)-(3.4) exist and are continuous in their respective domains, the scattering coefficients $T^\pm(k)$, $R^\pm(k)$, and $L^\pm(k)$ do not necessarily exist or are not necessarily continuous for all $k\in\mathbf{R}$. When they exist, we have

$$\mathbf{S}^\pm(-k) = \overline{\mathbf{S}^\pm(k)}, \qquad k\in\mathbf{R}.$$

Using the Jost solutions of (1.1) and (3.1), we obtain the Wronskian relations

$$[f_l^\pm(k,x);f_l^\mp(-k,x)] = -2ik = -2ik\frac{1-L^\pm(k)L^\mp(-k)}{T^\pm(k)\,T^\mp(-k)},$$

$$[f_r^\pm(k,x);f_r^\mp(-k,x)] = 2ik = 2ik\frac{1-R^\pm(k)R^\mp(-k)}{T^\pm(k)\,T^\mp(-k)}.$$

Contrary to the unitarity of $\mathbf{S}^{[0]}(k)$, the matrices $\mathbf{S}^{\pm}(k)$ are in general not unitary, but instead, for $k \in \mathbf{R}$ except at the singularities of $\mathbf{S}^{\pm}(k)$ as indicated in Theorem 3.1, we have

$$\mathbf{S}^{\pm}(k)\,\overline{\mathbf{S}^{\mp}(k)^t} = \mathbf{I},$$

where $\mathbf{I}$ is the 2×2 unit matrix and the superscript t denotes the matrix transpose. For such k we also have

$$\det \mathbf{S}^{\pm}(k) = T^{\pm}(k)^2 - L^{\pm}(k)R^{\pm}(k) = \frac{T^{\pm}(k)}{T^{\mp}(-k)}.$$

Theorem 3.1. *Assume $P,Q \in L^1(\mathbf{R})$. Then:*

(a) The functions $1/T^{\pm}(k)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$, their zeros in $\mathbf{C}^+$ are all isolated and can only accumulate on the real axis. The transmission coefficients $T^{\pm}(k)$ cannot have any zeros in $\overline{\mathbf{C}^+}\setminus\{0\}$.

(b) The quantities $L^{\pm}(k)/T^{\pm}(k)$ and $R^{\pm}(k)/T^{\pm}(k)$ are continuous on $\mathbf{R}\setminus\{0\}$.

(c) For fixed $k_0 \in \mathbf{R}\setminus\{0\}$, the quantities $1/T^{+}(k_0)$ and $R^{+}(k_0)/T^{+}(k_0)$ cannot be zero simultaneously; similarly, $1/T^{+}(k_0)$ and $L^{+}(k_0)/T^{+}(k_0)$ cannot be zero simultaneously. If $1/T^{+}(k_0)=0$ for some $k_0 \in \mathbf{R}\setminus\{0\}$, then the quantities $R^{\pm}(k_0)/T^{\pm}(k_0)$, $L^{\pm}(k_0)/T^{\pm}(k_0)$, $R^{\pm}(-k_0)/T^{\pm}(-k_0)$, $L^{\pm}(-k_0)/T^{\pm}(-k_0)$ are all nonzero.

(d) $R^{+}(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$ if and only if $1/T^{+}(k)$ does not vanish for $k \in \mathbf{R}\setminus\{0\}$. Similarly, $L^{+}(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$ if and only if $1/T^{+}(k)$ does not vanish for $k \in \mathbf{R}\setminus\{0\}$.

(e) For $k \in \mathbf{R}\setminus\{0\}$, the quantity $T^{+}(k)$ is continuous if and only if $R^{+}(k)$ is continuous; equivalently, $T^{+}(k)$ is continuous if and only if $L^{+}(k)$ is continuous.

(f) If, in addition, $Q \in L^1_1(\mathbf{R})$, then $k/[(k+i)\,T^{\pm}(k)]$ are continuous and bounded in $\overline{\mathbf{C}^+}$, and $kL^{\pm}(k)/T^{\pm}(k)$ and $kR^{\pm}(k)/T^{\pm}(k)$ are continuous and bounded on $\mathbf{R}$.

Proposition 3.2. *Assume $P(x) \le 0$ and $P,Q \in L^1(\mathbf{R})$. Then $\mathbf{S}^{+}(k)$ is continuous on $\mathbf{R}\setminus\{0\}$. If we further assume $Q \in L^1_1(\mathbf{R})$, then $\mathbf{S}^{+}(k)$ is continuous on $\mathbf{R}$.*

Theorem 3.3. *Assume $P,Q \in L^1(\mathbf{R})$. The scattering coefficients satisfy*

$$\frac{1}{|T^{\pm}(k)|^2} = 1 + \left|\frac{L^{\pm}(k)}{T^{\pm}(k)}\right|^2 \mp \int_{-\infty}^{\infty} dx\, |f_l^{\pm}(k,x)|^2 P(x), \qquad k \in \mathbf{R}\setminus\{0\},$$

$$\frac{1}{|T^{\pm}(k)|^2} = 1 + \left|\frac{R^{\pm}(k)}{T^{\pm}(k)}\right|^2 \mp \int_{-\infty}^{\infty} dx\, |f_r^{\pm}(k,x)|^2 P(x), \qquad k \in \mathbf{R}\setminus\{0\}.$$

Hence, if $P(x) \le 0$, $1/T^{+}(k)$ cannot have any zeros for $k \in \mathbf{R}$, and we have

$$|T^{+}(k)|^2 + |L^{+}(k)|^2 \le 1, \quad |T^{+}(k)|^2 + |R^{+}(k)|^2 \le 1, \qquad k \in \mathbf{R}.$$

If $1/T^{+}(k)$ does not have any zeros for $k \in \mathbf{R}$ and $P(x) \ge 0$, then we have

$$|T^{+}(k)|^2 + |L^{+}(k)|^2 \ge 1, \quad |T^{+}(k)|^2 + |R^{+}(k)|^2 \ge 1, \qquad k \in \mathbf{R}.$$

Corollary 3.4. *Assume $P,Q \in L^1(\mathbf{R})$ and $P(x) \le 0$. Then, for $k \in \mathbf{R}\setminus\{0\}$, we have $\frac{1}{|T^{+}(k)|^2} \ge 1$, and hence $1/T^{+}(k)$ cannot vanish on $\mathbf{R}$. Moreover, for $k \in \mathbf{R}\setminus\{0\}$, we have $\frac{1}{|T^{+}(k)|} \ge \frac{1}{|T^{-}(k)|}$.*

4. ASYMPTOTICS OF SCATTERING COEFFICIENTS

The large-k asymptotics of $\mathbf{S}^{\pm}(k)$ are summarized in the following theorem.

Theorem 4.1. *Assume $P, Q \in L^1(\mathbf{R})$. Then*

$$\frac{1}{T^{\pm}(k)} \exp\left(\pm\frac{1}{2}\int_{-\infty}^{\infty} dx P(x)\right) = 1 + o(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+},$$

$$\frac{R^{\pm}(k)}{T^{\pm}(k)} = o(1), \quad \frac{L^{\pm}(k)}{T^{\pm}(k)} = o(1), \qquad k \to \pm\infty.$$

Corollary 4.2. *Assume $P, Q \in L^1(\mathbf{R})$. If $1/T^{\pm}(k)$ does not vanish for $k \in \mathbf{R}$, then its number of zeros in $\mathbf{C}^+$ is finite. This occurs, in particular, if $P(x) \leq 0$.*

Next, we analyze the small-k asymptotics of $\mathbf{S}^{\pm}(k)$ in the generic and exceptional cases separately. In the exceptional case, we will see that $\mathbf{S}^{+}(0)$ is not determined by $Q(x)$ alone and obtain $\mathbf{S}^{+}(0)$ explicitly in terms of $P(x)$ and $Q(x)$.

Theorem 4.3. *Assume $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$ and suppose that we are in the generic case. Then $R^{\pm}(0) = L^{\pm}(0) = -1$, $T^{\pm}(k)$ vanish linearly as $k \to 0$ in $\overline{\mathbf{C}^+}$, and*

$$\lim_{k\to 0} \frac{2ik}{T^{+}(k)} = \lim_{k\to 0} \frac{2ik}{T^{-}(k)} = \lim_{k\to 0} \frac{2ik}{T^{[0]}(k)}.$$

Furthermore, $\det \mathbf{S}^{\pm}(0) = -1$*, and we have*

$$T^{\pm}(k) = \frac{2ik}{\int_{-\infty}^{\infty} dy Q(y) f_l^{[0]}(0,y)} + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}.$$

Theorem 4.4. *In the exceptional case, under the assumptions $P, Q \in L^1_1(\mathbf{R})$, we have*

$$T^{\pm}(0) = \frac{2\gamma}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx P(x) f_l^{[0]}(0,x)^2}, \tag{4.1}$$

$$L^{\pm}(0) = \frac{\gamma^2 - 1 \pm \int_{-\infty}^{\infty} dx P(x) f_l^{[0]}(0,x)^2}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx P(x) f_l^{[0]}(0,x)^2},$$

$$R^{\pm}(0) = \frac{1 - \gamma^2 \pm \int_{-\infty}^{\infty} dx P(x) f_l^{[0]}(0,x)^2}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx P(x) f_l^{[0]}(0,x)^2},$$

where γ is the constant defined in (2.6).

Theorem 4.5. *Let $Q \in L^1_1(\mathbf{R})$, and assume that $P \in L^1(\mathbf{R})$ in the generic case and $P \in L^1_1(\mathbf{R})$ in the exceptional case. Then:*

(a) *If any one of* $1/T^+(k)$, $R^+(k)/T^+(k)$, $L^+(k)/T^+(k)$ *is continuous at* $k=0$, *then all three are continuous at* $k=0$. *Moreover, either* $1/T^+(k)$ *and* $1/T^-(k)$ *are both continuous at* $k=0$ *or both discontinuous at* $k=0$.

(b) *In the generic case the six quantities* $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$, *and* $L^{\pm}(k)/T^{\pm}(k)$ *are all discontinuous at* $k=0$; *in the exceptional case, these quantities are all continuous at* $k=0$.

(c) *In the exceptional case* $1/T^+(k)$ *vanishes at* $k=0$ *if and only if* $\int_{-\infty}^{\infty} dx\, P(x) f_l^{[0]}(0,x)^2$ *is equal to* γ^2+1, *where* γ *is the constant defined in (2.6). In the generic case,* $k/T^+(k)$ *has a nonzero limit as* $k\to 0$.

(d) *Either the three quantities* $T^+(k)$, $R^+(k)$, $L^+(k)$ *are all continuous on* $\mathbf{R}$, *or they are all discontinuous on* $\mathbf{R}$.

In the special situation when $Q(x)=0$, we have $f_l^{[0]}(0,x)=f_r^{[0]}(0,x)=1$ and hence $\gamma=1$. This corresponds to the exceptional case. Using these values in (4.1) we see that

$$\frac{1}{T^+(0)}=1-\frac{1}{2}\int_{-\infty}^{\infty} dx\, P(x).$$

Hence, if $\int_{-\infty}^{\infty} dx\, P(x)=2$, no matter how smooth $P(x)$ is, we have $\frac{1}{T^+(0)}=0$ and $\frac{L^+(0)}{T^+(0)}=\frac{R^+(0)}{T^+(0)}=1$. In this case $\mathbf{S}^+(0)$ is clearly undefined.

5. BOUND STATES

Although $T^{[0]}(k)$ cannot have any singularities when $k\in\mathbf{R}$, we cannot rule out singularities of $T^+(k)$ when $k\in\mathbf{R}$ unless $P(x)\le 0$, as we have seen at the end of Section 4. Some other examples of such singularities will be presented in Section 8.

Theorem 5.1. *Assume* $P,Q\in L^1(\mathbf{R})$. *The zeros of* $1/T^+(k)$ *on the real axis do not correspond to the bound states of (1.1). Each zero of* $1/T^+(k)$ *in* $\mathbf{C}^+$ *corresponds to a bound state of (1.1). Conversely, if (1.1) has a bound state at some* $k_0\in\mathbf{C}^+$, *it is necessary that* $1/T^+(k_0)=0$.

Let us define

$$P_{\min}=\operatorname{ess}\inf_{x\in\mathbf{R}} P(x),\qquad P_{\max}=\operatorname{ess}\sup_{x\in\mathbf{R}} P(x),\qquad Q_{\min}=\operatorname{ess}\inf_{x\in\mathbf{R}} Q(x),\tag{5.1}$$

$$\beta^*=P_{\max}/2+\sqrt{P_{\max}^2/4-Q_{\min}}.$$

Note that, if $P,Q\in L^1(\mathbf{R})$, it follows that $P_{\max}\ge 0$ with equality holding if and only if $P(x)\le 0$, that $Q_{\min}\le 0$ with equality holding if and only if $Q(x)\ge 0$, and that $P_{\min}\le 0$ with equality holding if and only if $P(x)\ge 0$. Furthermore, $\beta^*\ge P_{\max}$ with equality holding if and only if $Q(x)\ge 0$.

Theorem 5.2. *Assume* $P,Q\in L^1(\mathbf{R})$, $P(x)\not\equiv 0$, *and* P_{max} *is finite. Then the zeros of* $1/T^+(k)$ *for* $P_{max}/2\le \operatorname{Im}k<\beta^*$ *can only occur on the imaginary axis, and all such zeros are simple. If, in addition,* $Q_{\min}$ *is finite, then there are no zeros of* $1/T^+(k)$ *in the region* $\{k\in\mathbf{C}^+ : (\operatorname{Im}k)^2-(\operatorname{Re}k)^2-(\operatorname{Im}k)P_{\max}\ge -Q_{\min}\}$. *Consequently,* $1/T^+(k)$ *has no zeros in* $\mathbf{C}^+$ *satisfying* $\operatorname{Im}k\ge\beta^*$.

Theorem 5.3. *Assume $Q(x) = 0$ and $P \in L^1(\mathbf{R})$. If $\int_{-\infty}^{\infty} dx P(x) > 2$, then (1.1) has at least one bound state at $k = i\beta$ for some positive β. If $\int_{-\infty}^{\infty} dx\,|P(x)| \leq 2$, then $1/T^+(k)$ has no zeros in $\mathbf{C}^+$.*

Theorem 5.4. *Assume that $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$. At the k-values in $\mathbf{C}^+$ satisfying $\int_{-\infty}^{\infty} dx\,|ikP(x) + Q(x)| \leq 2|k|$, there are no zeros of $1/T^+(k)$. Moreover, there are no zeros of $1/T^+(k)$ in $\mathbf{C}^+ \setminus \{i\kappa_1, \cdots, i\kappa_N\}$ satisfying $|T^{[0]}(k)|\,||P||_{1,1} < 2e^{-||Q||_{1,1}}$, where κ_j correspond to the bound states of (1.2).*

Next we analyze the change in $N(P,Q)$, the number of bound states of (1.1), when we perturb $P(x)$ or $Q(x)$. In the next two theorems we write $T^+(k;P,Q)$ for the transmission coefficient of (1.1) to emphasize its dependence on $P(x)$ and $Q(x)$. By $||f||_1$ we denote the norm on $L^1(\mathbf{R})$, i.e. $||f||_1 = \int_{-\infty}^{\infty} dx\,|f(x)|$.

Theorem 5.5. *Assume $P_1, P_2 \in L^1(\mathbf{R})$ and $Q_1, Q_2 \in L^1_1(\mathbf{R})$, and suppose $1/T^+(k;P_1,Q_1)$ does not have any real zeros and $Q_1(x)$ is a generic potential. If $||P_1 - P_2||_1 + ||Q_1 - Q_2||_{1,1}$ is small enough, then*

(a) $1/T^+(k;P_2,Q_2)$ does not have any real zeros.

(b) $N(P_2,Q_2) = N(P_1,Q_1)$.

(c) If all zeros of $1/T^+(k;P_1,Q_1)$ are simple and purely imaginary, then the zeros of $1/T^+(k;P_2,Q_2)$ are also simple and purely imaginary.

Theorem 5.6. *Assume $P_1, P_2, Q \in L^1_1(\mathbf{R})$, $1/T^+(k;P_1,Q)$ does not have any real zeros, and $Q(x)$ is an exceptional potential. If $||P_1 - P_2||_{1,1}$ is small enough, then*

(a) $1/T^+(k;P_2,Q)$ does not have any real zeros.

(b) $N(P_2,Q) = N(P_1,Q)$.

(c) If all zeros of $1/T^+(k;P_1,Q)$ are simple and purely imaginary, then the zeros of $1/T^+(k;P_2,Q)$ are also simple and purely imaginary.

When $P(x) \leq 0$, we can say more about the bound states of (1.1). From Theorem 5.2 we get the following:

Corollary 5.7. *Assume $P(x) \leq 0$ and $P,Q \in L^1(\mathbf{R})$. Then, the poles of $T^+(k)$ in $\mathbf{C}^+$ are all purely imaginary and simple. In addition, assume that $Q_{\min}$ defined in (5.1) is finite; then there are no zeros of $1/T^+(k)$ in $\overline{\mathbf{C}^+}$ for $\operatorname{Im} k \geq \sqrt{-Q_{\min}}$. In particular, if $P(x) \leq 0$ and $Q(x) \geq 0$, then $1/T^+(k)$ has no zeros in $\overline{\mathbf{C}^+}$.*

When $P(x) \leq 0$, under additional assumptions on $P'(x)$, Pivovarchik has shown that [5] the number of bound states of the radial analog of (1.1) is independent of $P(x)$ and that [6–8] the bound states can only occur when k is located on the positive imaginary axis in $\mathbf{C}^+$ and each bound state is simple. The results were actually obtained for a class of abstract operator polynomials with the radial analog of (1.1) as an example. It is possible to obtain Pivovarchik's results on the full line and without assuming the differentiability of $P(x)$.

Theorem 5.8. *Assume $P,Q \in L^1(\mathbf{R})$ and $P(x) \leq 0$. If $N(0,Q) = +\infty$, then we also have $N(P,Q) = +\infty$. If $N(0,Q)$ is finite, then we have $N(P,Q) = N(0,Q)$. Thus, the number of bound states of (1.1) coincides with the number of bound states of (1.2).*

Theorem 5.9. *Assume $N(0,Q)$ is finite and nonzero, $P(x) \le 0$, and suppose $P,Q \in L^1(\mathbf{R})$ and P_{min} is finite, where P_{min} is the constant defined in (5.1). Let $k = i\kappa_j$ correspond to the bound states of (1.2) for $j = 1, \cdots, \mathcal{N}$. Then, the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ occur at $k = i\beta_j$ satisfying $\beta_* \le \beta_j \le \kappa_j$ for $j = 1, \cdots, \mathcal{N}$, where $\beta_* = P_{min}/2 + \sqrt{P_{min}^2/4 + \kappa_1^2}$. In particular, $\beta_1 \ge \beta_*$ and $\beta_{\mathcal{N}} \le \kappa_{\mathcal{N}}$, with equalities holding if and only if $P(x) \equiv 0$.*

Theorem 5.10. *Assume $P,Q \in L^1(\mathbf{R})$, $P(x) \le 0$, and $N(0,Q) = +\infty$, and let $\{\mathcal{E}_j\}$ and $\{\mathcal{E}_j^{[0]}\}$ for $j \ge 1$ denote the bound-state energies of (1.1) and (1.2), respectively, ordered such that $\mathcal{E}_j < \mathcal{E}_{j+1}$ and $\mathcal{E}_j^{[0]} < \mathcal{E}_{j+1}^{[0]}$. Then, we have $\mathcal{E}_j^{[0]} \le \mathcal{E}_j < 0$ for $j \ge 1$, and hence the bound-state energies of (1.1) cannot occur below the lowest bound-state energy of (1.2).*

Recall that the Levinson theorem [9, 10] relates the number of bound states of the Schrödinger equation to the change in the phase of the transmission coefficient. Next we present an analog of the Levinson theorem for (1.1).

Theorem 5.11. *Assume that $P \in L^1(\mathbf{R})$ in the generic case and $P \in L_1^1(\mathbf{R})$ in the exceptional case and that $Q \in L_1^1(\mathbf{R})$, and suppose $1/T^+(k)$ does not have any real zeros. Then the number of bound states of (1.1) is given by*

$$N(P,Q) = \frac{d}{2} + \frac{1}{\pi}\arg T^+(0+),$$

where $d = 0$ in the exceptional case and $d = 1$ in the generic case, and $\arg T^+(k)$ *denotes the continuous branch of the argument of $T^+(k)$ normalized so that* $\arg T^+(+\infty) = 0$.

6. EIGENVALUE BRANCHES

In this section we consider the bound states of (1.1) when k is on the positive imaginary axis in $\mathbf{C}^+$; in other words, we consider the negative-energy bound states of (1.1). As indicated in Theorem 5.1, (1.1) cannot have any bound states at zero or positive energies. When $P(x) \le 0$, as seen in Theorem 5.9, the bound states of (1.1) can only occur at negative energies. However, unless $P(x) \le 0$, there may exist also bound states at complex energies, some examples of which will be given in Section 8.

The negative-energy bound states of (1.1) can be analyzed in terms of the eigenvalue curves of the differential operator $\mathcal{O}_\beta$ given by

$$\mathcal{O}_\beta = -d^2/dx^2 + Q(x) - \beta P(x). \tag{6.1}$$

Let us write (1.1) when $k = i\beta$ as a system of two simultaneous equations:

$$-\psi'' + V(\beta,x)\,\psi = E(\beta)\,\psi, \tag{6.2}$$

$$E(\beta) = -\beta^2, \tag{6.3}$$

where β is considered to be a parameter in the potential $V(\beta,x) = Q(x) - \beta P(x)$ of the Schrödinger equation (6.2), and $E(\beta)$ denotes the corresponding energy for each β. Each bound-state energy of (1.2) gives rise to an eigenvalue branch $E(\beta)$ of $\mathcal{O}_\beta$. Note that for each $\beta > 0$, a nontrivial solution of (1.1) belonging to $L^2(\mathbf{R})$ corresponds to an eigenvector of $\mathcal{O}_\beta$

with the eigenvalue E, which we write $E(\beta)$ to emphasize its dependence upon β. Thus, we see that the negative-energy bound states of (1.1) correspond to the eigenvalues $E(\beta)$ that intersect the parabola $E = -\beta^2$ in the (β,E)-plane. Assume that the eigenvalue branch $E(\beta)$ and the parabola $E = -\beta^2$ intersect at $(\beta_0, -\beta_0^2)$. Then, β_0 corresponds to a bound state of (1.1) with negative energy. Conversely, any negative bound-state energy of (1.1) can be identified with a simultaneous solution of (6.2) and (6.3). Hence, we can analyze the negative-energy bound states of (1.1) by analyzing the eigenvalue curves of $\mathcal{O}_\beta$ and their intersections with the parabola $E = -\beta^2$.

Associated with the eigenvalue $E_0(\beta)$ there exists [11] a real-valued, analytic eigenvector $\psi(\beta,x)$. Near $\beta = \beta_0$ we have the convergent expansions

$$E_0(\beta) = \sum_{n=0}^{\infty} a_n (\beta - \beta_0)^n, \qquad \psi(\beta,x) = \sum_{n=0}^{\infty} \psi_n(x) (\beta - \beta_0)^n, \tag{6.4}$$

with $\psi_n \in L^2(\mathbf{R})$ for $n \geq 0$. One may choose $\psi_0(x) = f_l^+(i\beta_0,x)$. We can recursively determine a_n and $\psi_n(x)$. In fact, we have

$$a_0 = E_0(\beta_0), \qquad a_1 = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} dx P(x) \psi_0(x)^2, \tag{6.5}$$

$$a_n = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} dx \, \psi_0(x) \left(P(x) \psi_{n-1}(x) + \sum_{j=1}^{n-1} a_j \psi_{n-j}(x) \right), \qquad n \geq 2.$$

For the lowest eigenvalue, one obtains

$$a_2 = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} \frac{dx}{\psi_0(x)^2} \left(\int_{-\infty}^{x} dt \, \psi_0(t)^2 [P(t) + a_1] \right)^2.$$

Theorem 6.1. *Suppose $P,Q \in L^1(\mathbf{R})$. Then, the lowest eigenvalue branch satisfies $E''(\beta) \leq 0$ for $\beta > 0$ with equality holding if and only if $P(x) \equiv 0$.*

An eigenvalue curve $E(\beta)$ may cut the parabola $E = -\beta^2$ at two or more points, and if this happens each intersection gives rise to a negative-energy bound state of (1.1). Moreover, $E(\beta) + \beta^2$ may have double or higher-order zeros; then, the order of the zero of $E(\beta) + \beta^2$ is the same as the multiplicity of the corresponding bound state.

Theorem 6.2. *Suppose $P,Q \in L^1(\mathbf{R})$. Then, $1/T^+(i\beta)$ has a zero of order m at some positive β_0 if and only if the function $E_0(\beta) + \beta^2$ has a zero of order m at β_0, where $E_0(\beta)$ denotes the unique eigenvalue branch of the operator $\mathcal{O}_\beta$ satisfying $E_0(\beta) \to -\beta_0^2$ as $\beta \to \beta_0$. Moreover, if (1.2) has $N(0,Q)$ bound states, then (1.1) has at least $N(0,Q)$ bound states with negative energies.*

If β_0 corresponds to a zero of $E_0(\beta) + \beta^2$ of order m for some $m \geq 1$, then the coefficients a_n in (6.4) are determined for $n = 0, 1, \cdots, m-1$ by expanding $E_0(\beta) + \beta^2$ about β_0. Thus, for $m = 1$ we get $a_0 = -\beta_0^2$; for $m = 2$ we have $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$; for $m = 3$ we get $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$, $a_2 = -1$; for $m \geq 4$ we get $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$, $a_2 = -1$, and $a_3 = \cdots = a_{m-1} = 0$.

If $P(x) \leq 0$, from (6.5) we see that $a_1 \geq 0$ for any positive β_0 with equality holding if and only if $P(x) \equiv 0$. Thus, when $P(x) \leq 0$ we have $E'(\beta) \geq 0$, and as a result each eigenvalue branch $E_j(\beta)$ is a nondecreasing function of β. Therefore, for $\beta > 0$, the graph of each eigenvalue branch $E_j(\beta)$ intersects the parabola $E = -\beta^2$ at exactly one point, say $(\beta_j, -\beta_j^2)$, and each $E_j(\beta)$ gives rise to exactly one solution of (6.3). Hence, there is a one-to-one correspondence between the bound states of (1.1) and the bound states of (1.2), and $E_j(\beta)$ satisfies $E_j(0) = -\kappa_j^2$. The number $N(P,Q)$ is equal to the number of intersections of the parabola in (6.3) with the eigenvalue branches $E_j(\beta)$ for $j \geq 1$. Since each of the $N(0,Q)$ branches is responsible for exactly one intersection, we conclude that $N(P,Q) = N(0,Q)$. Note that if $Q \in L^1(\mathbf{R})$ but $Q \notin L^1_1(\mathbf{R})$, it is possible that $N(0,Q) = +\infty$, but then we also have $N(P,Q) = +\infty$.

7. ZEROS OF JOST SOLUTIONS

In this section we study the zeros of the Jost solutions of (1.1) for fixed k and analyze the number of such zeros in relation to the bound states of (1.1) and (1.2).

Concerning the zeros of the Jost solutions of (1.2) when k is on the positive imaginary axis in $\mathbf{C}^+$, the following is already known [12, 13]:

Proposition 7.1. *Suppose $Q \in L^1(\mathbf{R})$ and $\beta > 0$. Then the number of zeros of $f_l^{[0]}(i\beta,x)$ is equal to the number of bound states of (1.2) with energies contained in the interval $(-\infty, -\beta^2)$. Suppose further that $Q \in L^1_1(\mathbf{R})$. Then, the number of zeros of $f_l^{[0]}(0,x)$ is equal to $N(0,Q)$.*

The next proposition concerns the zeros of the Jost solutions when k lies off the positive imaginary axis in $\mathbf{C}^+$.

Proposition 7.2. *Assume $P,Q \in L^1(\mathbf{R})$ and $k \in \mathbf{C}^+$. If $P(x) \leq 2\,\mathrm{Im}\,k$, then $f_l^+(k,x)$ and $f_r^+(k,x)$ cannot vanish for any $x \in \mathbf{R}$.*

When k is confined to the positive imaginary axis, one can analyze the zeros of the Jost solutions of (1.1) by using the methods [13, 14] developed for (1.2). At a fixed nonnegative β, one can show that $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros. These zeros are simple, and they are interlaced when $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ are linearly independent.

Theorem 7.3. *Suppose $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$, and assume that (1.1) has a bound state of multiplicity m at $k = i\beta_0$ for some positive β_0. Then, the number of zeros of $f_l^+(i\beta,x)$ behaves in the following manner as β is increased from $\beta_0 - \epsilon$ to $\beta_0 + \epsilon$ for sufficiently small and positive ϵ: If m is even, then the number of zeros is either constant throughout the interval $(\beta_0 - \epsilon, \beta_0 + \epsilon)$ or it is constant in $(\beta_0 - \epsilon, \beta_0) \cup (\beta_0, \beta_0 + \epsilon)$ but one less at β_0. If m is odd, then the number of zeros either increases or decreases by one as β crosses β_0. The number of zeros of $f_l^+(i\beta,x)$ can only change at β-values corresponding to the bound states of (1.1).*

When $P(x) \leq 0$, the number of zeros of the Jost solutions of (1.1) when k is on the positive imaginary axis in $\mathbf{C}^+$ is related to the bound states in a simple manner.

Theorem 7.4. *Assume that $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, and $P(x) \leq 0$. Then, for each $\beta \geq 0$, the functions $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros, and this number is equal to the number of bound states of (1.1) with energies contained in the interval $(-\infty, -\beta^2)$.*

For further results on the zeros of the Jost solutions of (1.1), we refer the reader to [1].

8. EXAMPLES

In this section we present explicitly solved examples illustrating the theory presented in the earlier sections. The numerical values in these examples were obtained by using the mathematical software Maple.

Our first example shows that if we relax the condition $P \in L^1(\mathbf{R})$, the scattering matrix $\mathbf{S}^+(k)$ may not exist at all.

Example 8.1. *Let $P(x)$ and $Q(x)$ have support in $(0,+\infty)$ and be given by*

$$Q(x) = \theta(x)\frac{2}{(1+x)^2}, \qquad P(x) = \theta(x)\frac{2}{1+x}, \tag{8.1}$$

where $\theta(x)$ is the Heaviside function; thus $P \notin L^1(\mathbf{R})$. Two linearly independent solutions of (1.1) are given by

$$\psi_1^+(k,x) = \theta(x)\frac{e^{-ikx}}{1+x} + \theta(-x)\left[e^{-ikx} - \frac{\sin kx}{k}\right],$$

$$\psi_2^+(k,x) = \theta(x)\left[x+1+\frac{i}{k} - \frac{1}{2k^2(1+x)}\right]e^{ikx} + \theta(-x)F(k,x),$$

where we have defined

$$F(k,x) = \left(\frac{1}{k} + \frac{1}{2k^3}\right)\sin kx + \left(1 + \frac{i}{k} - \frac{1}{2k^2}\right)e^{ikx}.$$

Note that $\psi_1^+(k,x) \to 0$ and $\psi_2^+(k,x) = O(x)$ as $x \to +\infty$; hence, we cannot form a solution of (1.1) asymptotic to e^{ikx} as $x \to +\infty$. Although we can form a linear combination of $\psi_1^+(k,x)$ and $\psi_2^+(k,x)$ that is asymptotic to e^{-ikx} as $x \to -\infty$, the resulting function is not bounded as $x \to +\infty$. Thus, there are no scattering solutions and no scattering matrices corresponding to the potentials given in (8.1). Note that the scattering matrix $\mathbf{S}^{[0]}(k)$ corresponding to (1.2) with $Q(x)$ given in (8.1) is well defined, and we have

$$T^{[0]}(k) = \frac{2k^2}{2k^2+2ik-1}, \qquad L^{[0]}(k) = -R^{[0]}(k) = \frac{1}{2k^2+2ik-1}.$$

Contrary to the case $P(x) = 0$, the scattering matrix $\mathbf{S}^+(k)$ is in general not determined if one of the reflection coefficients and the bound state energies are known, as illustrated by the following example.

Example 8.2. *Assume $Q \in L_1^1(\mathbf{R})$ is an exceptional potential without bound states. Let*

$$P(x) = 2\frac{f_l^{[0]\prime}(0,x)}{f_l^{[0]}(0,x)}, \tag{8.2}$$

where $f_l^{[0]}(0,x)$ is the zero-energy Jost solution of (1.2); note that $f_l^{[0]}(0,x)$ is uniquely determined by $Q(x)$ alone and $P(x)$ given in (8.2) necessarily belongs to $L^1(\mathbf{R})$. The corresponding scattering matrices $\mathbf{S}^{\pm}(k)$ can be evaluated explicitly, and we have

$$T^+(k) = \frac{1}{\gamma}, \quad L^+(k) = 0, \quad R^+(k) = 2\int_{-\infty}^{\infty} dy\, \frac{f_l^{[0]\prime}(0,y)}{f_l^{[0]}(0,y)^3}e^{-2iky}, \tag{8.3}$$

$$T^{-}(k)=\gamma, \quad R^{-}(k)=0, \quad L^{-}(k)=-2\gamma^2\int_{-\infty}^{\infty}dy\,\frac{f_l^{[0]\prime}(0,y)}{f_l^{[0]}(0,y)^3}e^{2iky},$$

where γ is the constant defined in (2.6). As seen from (8.3), $T^{+}(k)$ and $L^{+}(k)$ cannot determine $R^{+}(k)$, and there are infinitely many $R^{+}(k)$ corresponding to these two scattering coefficients. Therefore, the coefficients $P(x)$ and $Q(x)$ cannot in general be determined from the scattering data consisting of the transmission coefficient and of only one of the reflection coefficients. Note also that $\mathbf{S}^{+}(k)$ is in general not determined by only one or two of its entries.

In the next example we show that $1/T^{+}(k)$ may have zeros on $\mathbf{R}$ or off the positive imaginary axis in $\mathbf{C}^{+}$. We also consider the zeros of $1/T^{+}(k)$ on the positive imaginary axis and illustrate the fact that unless $P(x)\le 0$, the number of negative-energy bound states of (1.1) may be more than the number of bound states of (1.2).

Example 8.3. *For real parameters a and b, let*

$$P(x)=\begin{cases} b, & x\in(0,1),\\ 0, & \textit{elsewhere},\end{cases} \qquad Q(x)=\begin{cases} a, & x\in(0,1),\\ 0, & \textit{elsewhere}.\end{cases} \tag{8.4}$$

The resulting transmission coefficient can be obtained explicitly and we have

$$\frac{1}{T^{+}(k)}=e^{ik}\left[\cos s+\frac{k^2+s^2}{2iks}\sin s\right],$$

where we have defined $s=\sqrt{k^2-ibk-a}$. Let us use an overline on the last digit to indicate a roundoff. When $a=-9.273\overline{8}$ and $b=3.970\overline{8}$, we find simple zeros of $1/T^{+}(k)$ at $k=\pm 1$. When $a=0$ we have

$$\frac{1}{T^{+}(0)}=1-\frac{b}{2}, \qquad \frac{L^{+}(0)}{T^{+}(0)}=\frac{R^{+}(0)}{T^{+}(0)}=\frac{b}{2},$$

and hence $1/T^{+}(0)=0$ if $b=2$. When $b=0$ and $a<0$, we have a square-well potential, and in this case (1.1) has $\mathcal{N}$ bound states such that

$$(\mathcal{N}-1)\pi<\sqrt{-a}\le\mathcal{N}\pi. \tag{8.5}$$

When $a=-100$ and $b=0$, from (8.5) we see that we get four bound states of (1.1) at $k=i\kappa_j$ with

$$\kappa_1=1.9\overline{3}, \quad \kappa_2=6.4\overline{1}, \quad \kappa_3=8.5\overline{5}, \quad \kappa_4=9.6\overline{5}. \tag{8.6}$$

When $a=-100$ and $b=-10$, there are four bound states at $k=i\beta_j$, where

$$\beta_1=0.7\overline{6}, \quad \beta_2=3.5\overline{5}, \quad \beta_3=5.1\overline{1}, \quad \beta_4=5.9\overline{2}. \tag{8.7}$$

When $a=-100$ and $b=-100$, there are still four bound states with

$$\beta_1=0.1\overline{1}, \quad \beta_2=0.5\overline{8}, \quad \beta_3=0.8\overline{6}, \quad \beta_4=0.9\overline{7}. \tag{8.8}$$

From (8.6)-(8.8), we see that as b becomes more negative the bound-state energies are pushed toward zero. Now let us see what happens when $b>0$. By Theorem 5.3, if $a=0$ and $b>2$, we must have a bound state at $k=i\beta$ for some positive β. Letting $a=0$ and $b=21/10$, we obtain a bound state at $k = 0.1\overline{5}i$; from the plot of $T^+(k)$ for $k \in (0, +\infty)$, we see that this is the only bound state. Note that when $b > 0$ we cannot exclude the possibility of bound states with k off the imaginary axis in $\mathbf{C}^+$. For example, when $a = -93/10$ and $b = 4$, we find bound states at $k = \pm 0.976\overline{4} + 0.023\overline{3}i$; in this case there are no other bound states. Choosing $a = 0$ and $b = 10$, we obtain over 200 bound states, only three of which correspond to the k-values on the positive imaginary axis with $k = i\beta_j$, where

$$\beta_1 = 2.1\overline{4}, \quad \beta_2 = 5.9\overline{6}, \quad \beta_3 = 9.2\overline{7}. \tag{8.9}$$

Choosing $a = 0$ and $b = 100$, when k is on the positive imaginary axis we obtain thirty-one bound states with

$$\begin{array}{llll}
\beta_1 = 0.1\overline{0}, & \beta_2 = 0.4\overline{1}, & \beta_3 = 0.9\overline{3}, & \beta_4 = 1.6\overline{7}, \\
\beta_5 = 2.6\overline{4}, & \beta_6 = 3.8\overline{5}, & \beta_7 = 5.3\overline{3}, & \beta_8 = 7.0\overline{9}, \\
\beta_9 = 9.1\overline{9}, & \beta_{10} = 11.6\overline{9}, & \beta_{11} = 14.6\overline{3}, & \beta_{12} = 18.2\overline{0}, \\
\beta_{13} = 22.6\overline{1}, & \beta_{14} = 28.4\overline{3}, & \beta_{15} = 37.6\overline{3}, & \beta_{16} = 60.4\overline{1}, \\
\beta_{17} = 69.6\overline{9}, & \beta_{18} = 75.6\overline{0}, & \beta_{19} = 80.1\overline{1}, & \beta_{20} = 83.7\overline{7}, \\
\beta_{21} = 86.8\overline{3}, & \beta_{22} = 89.4\overline{2}, & \beta_{23} = 91.6\overline{3}, & \beta_{24} = 93.5\overline{2}, \\
\beta_{25} = 95.1\overline{2}, & \beta_{26} = 96.4\overline{6}, & \beta_{27} = 97.5\overline{7}, & \beta_{28} = 98.4\overline{6}, \\
\beta_{29} = 99.1\overline{4}, & \beta_{30} = 99.6\overline{2}, & \beta_{31} = 99.9\overline{1}, &
\end{array}$$

and there are also many more bound states corresponding the k-values off the imaginary axis in $\mathbf{C}^+$. Note that the bound states may occur even when $a > 0$ and $b > 0$. For example, when $a = 1$ and $b = 10$, we obtain many bound states, four of which correspond to the k-values on the positive imaginary axis with $k = i\beta_j$, where

$$\beta_1 = 0.1\overline{3}, \quad \beta_2 = 2.5\overline{0}, \quad \beta_3 = 5.6\overline{3}, \quad \beta_4 = 9.1\overline{6}.$$

Next we present an example where $T^+(k)$ has a double pole on the positive imaginary axis in $\mathbf{C}^+$.

Example 8.4. *Let*

$$P(x) = \frac{4b\epsilon c e^{-2\epsilon|x|}}{1 + ce^{-2\epsilon|x|}}, \qquad Q(x) = \frac{4\epsilon^2 ce^{-2\epsilon|x|}[-3b - 2 + b^2 ce^{-2\epsilon|x|}]}{(1 + ce^{-2\epsilon|x|})^2},$$

with $\epsilon > 0$, $c \in (-1, -5 + \sqrt{20})$, *and* $b \in \mathbf{R}$. *The transmission coefficient can be evaluated explicitly, and we have*

$$T^+(k) = \frac{k(k + i\epsilon)^2(1 + c)^{2b}}{(k - k_0)(k - k_+)(k - k_-)}, \tag{8.10}$$

where we have defined

$$k_0 = i\frac{\epsilon}{1 + c}[-1 + c + 2bc],$$

$$k_\pm = \frac{i\epsilon}{2(1 + c)}\left[(-1 + c + 4bc) \pm \sqrt{1 + c^2 + 14c + 16bc}\right].$$

When $b = -(c^2 + 14c + 1)/(16c)$, *we get* $k_\pm = -i\epsilon(c^2 + 10c + 5)/[8(1 + c)]$, *and hence* $T^+(k)$ *given in (8.10) has a double pole on the positive imaginary axis. When* $b = (1 - c)/(4c)$, *note that* k_0 *is located on the negative imaginary axis and that* k_+ *and* k_- *are symmetrically located on the real axis; thus, in this case* $T^+(k)$ *has poles on the real axis. When* $b = -(5 + \sqrt{5})/10$ *and* $c = -5 + \sqrt{20}$, *we get* $k_+ = k_- = 0$, *and hence* $T^+(k)$ *has a simple pole at* $k = 0$.

The next example concerns the zeros of $f_l^+(i\beta,x)$ when $\beta > 0$.

Example 8.5. *Consider the same $P(x)$ and $Q(x)$ as studied in Example 8.3. For the various specific values of a and b listed in that example, the zeros of $1/T^+(i\beta)$ are all simple. Hence, as Theorem 7.3 states, we expect the number of zeros of $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ to change by one at the zeros of $1/T^+(i\beta)$ as β varies in $(0,+\infty)$. For example, when $a = 0$ and $b = 10$, using β_1, β_2, and β_3 given in (8.9), one finds that $f_l^+(i\beta,x)$ has one zero for $\beta \in [0,\beta_1)$, two zeros for $\beta \in (\beta_1,\beta_2)$, one zero for $\beta \in (\beta_2,\beta_3)$, and no zeros for $\beta \in (\beta_3,+\infty)$. When $a = 0$ and $b = 21/10$, one finds that $f_l^+(i\beta,x)$ has one zero for $\beta \in [0,\beta_1)$ and no zeros for $\beta \in (\beta_1,+\infty)$, where $\beta_1 = 0.1\overline{5}$. When $a = 0$ and $b = 100$, one finds that $f_l^+(i\beta,x)$ has no zeros for $\beta \in (\beta_{31},+\infty)$, one zero for $\beta \in [0,\beta_1)$ and one zero for $\beta \in (\beta_{30},\beta_{31})$, j zeros for $\beta \in (\beta_{j-1},\beta_j)$ and j zeros for $\beta \in (\beta_{31-j},\beta_{32-j})$ with $j = 2,3,\cdots,15$, and sixteen zeros for $\beta \in (\beta_{16},\beta_{17})$. On the other hand, for $a = 19.85\overline{2}$ and $b = 10$, there is one negative-energy bound state of (1.1) of multiplicity two occurring at $k = i\beta_1$ with $\beta_1 = 4.72\overline{4}$; in this case $f_l^+(i\beta,x)$ has no zeros for any $\beta \geq 0$.*

9. INVERSE PROBLEMS

Inverse problems related to (1.1) consist of the recovery of $P(x)$ or $Q(x)$ from an appropriate set of scattering data. One such inverse problem is to recover both $P(x)$ and $Q(x)$. In the radial case, when there are no bound states, Jaulent and Jean presented [15] an inversion method when $Q(x)$ is real and $P(x)$ is imaginary. They also extended their method to solve the inverse problem on the full line for real $Q(x)$ and imaginary $P(x)$ [16, 17]. By this method, using the scattering data $\{R^+(k), R^-(k)\}$, one solves a pair of two coupled Marchenko integral equations, and these solutions are used in a first-order ordinary differential equation whose solution leads to $P(x)$. Jaulent [18] also extended this method to the case when $P(x)$ is real although complete details and proofs were not given. When $P(x)$ is purely imaginary and $\int_{-\infty}^{\infty} dz\, P(z) = 0$, Sattinger and Szmigielski [19] showed that one can simplify the method of Jaulent and Jean and recover $P(x)$ by solving an algebraic equation rather than a differential equation.

When $P(x)$ is purely imaginary, the methods available for self-adjoint differential operators can be employed to analyze the inverse scattering problem for (1.1); in this case the scattering matrices $\mathbf{S}^{\pm}(k)$ are unitary and the reflection coefficients cannot exceed one in absolute value. However, when $P(x)$ is real, the differential operator pertaining to (1.1) is no longer self-adjoint and the scattering matrices $\mathbf{S}^{\pm}(k)$ are no longer unitary. Consequently, the analysis of the direct and inverse scattering problems with real $P(x)$ is more difficult than with imaginary $P(x)$. As we have seen, for example, the non-self-adjointness of the differential operator in (1.1) may lead to singularities of the transmission coefficient on $\mathbf{R}$, and the reflection coefficients may not be bounded by one in absolute value and hence the Marchenko integral operators are in general not contractive. When $P(x) \leq 0$, some of the usual properties of the one-dimensional Schrödinger equation given in (1.2), such as the simplicity of the poles of the transmission coefficient, the confinement of these poles to the positive imaginary axis in $\mathbf{C}^+$, and the absence of singularities of the transmission coefficient at real-k values are still valid for (1.1). However, in the available inversion methods to recover $P(x)$ and $Q(x)$ one needs the scattering data associated with both (1.1) and (3.1). Hence, even when we study the inversion problem for absorptive media where one requires $P(x) \leq 0$, one may have to deal with bound-state scattering data for (3.1) which may involve complex-energy or degenerate bound states, unless the absorption is sufficiently weak.

Finally, let us mention the study by Kaup [20] on the direct and inverse scattering problem for

$$\phi'' + \left[k^2 + \frac{1}{4\beta^2}\right]\phi = [ikP(x) + Q(x)]\phi, \tag{9.1}$$

where β is a nonzero constant and $P, Q \in L_1^1(\mathbf{R})$. Under additional restrictions on $P(x)$, Tsutsumi [21] analyzed the scattering problem for (9.1) with $\beta = \frac{1}{2}$ by using a 2×2 matrix analog of (1.1) with k replaced by $\sqrt{k^2+1}$. Sattinger and Szmigielski [22] studied the direct and inverse scattering problem for (9.1) when $\beta = \frac{1}{2}$, $\int_{-\infty}^{\infty} dx P(x) = 0$, and $P(x)$ and $Q(x)$ are in the Schwartz space. The inverse scattering problem for (9.1) is used to solve an initial-value problem for a coupled system of two nonlinear evolution equations, and that inverse problem is analyzed by studying an associated Riemann–Hilbert problem [22]. We should emphasize that both the direct and inverse scattering problems for (1.1) are somewhat different from those for (9.1). The direct problem for (9.1) is analyzed in the complex-z plane using Kaup's transformations $k = \frac{1}{4}[2z - 1/(2\beta^2 z)]$ and $E = \sqrt{k^2+1} = \frac{1}{4}[2z + 1/(2\beta^2 z)]$.

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