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Wave scattering in one dimension with absorption

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Wave scattering is analyzed in a one-dimensional nonconservative medium governed by the generalized Schrödinger equation $d^2\psi/dx^2+k^2\psi=[ikP(x)+Q(x)]\psi$, where $P(x)$ and $Q(x)$ are real, integrable potentials with finite first moments. Various properties of the scattering solutions are obtained. The corresponding scattering matrix is analyzed, and its small-k and large-k asymptotics are established. The bound states, which correspond to the poles of the transmission coefficient in the upper-half complex plane, are studied in detail. When the medium is not purely absorptive, i.e., unless $P(x)\leqslant 0$, it is shown that there may be bound states at complex energies, degenerate bound states, and singularities of the transmission coefficient imbedded in the continuous spectrum. Some explicit examples are provided illustrating the theory.
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I. INTRODUCTION

Wave propagation in a one-dimensional nonconservative medium is described, in the frequency domain, by the generalized Schrödinger equation

$$\psi^{+\prime\prime}(k,x)+k^2\psi^{+}(k,x)=[ikP(x)+Q(x)]\psi^{+}(k,x),\quad x\in\mathbf{R}, \tag{1.1}$$

where $\mathbf{R}$ is the real line, the prime denotes the derivative with respect to the spatial coordinate x, k is the wave number (also known as the momentum), k^2 is the energy, $P(x)$ describes the combined effect of energy absorption and energy generation, and $Q(x)$ denotes the restoring force density. In the time domain, (1.1) corresponds to a wave equation of the form

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}-P(x)\,\frac{\partial u}{\partial t}=Q(x)u,\quad t,x\in\mathbf{R},$$

where the wave speed is equal to 1. When $P(x)\leqslant 0$, there is net absorption; however, unless otherwise stated we will not put any restriction on the sign of $P(x)$. In the sequel a significant role will be played by the associated equation

$$\psi^{-\prime\prime}(k,x)+k^2\psi^{-}(k,x)=[-ikP(x)+Q(x)]\psi^{-}(k,x),\quad x\in\mathbf{R}, \tag{1.2}$$

where the sign of $P(x)$ in (1.1) has been changed.

Let $L^p_q(I)$ denote the measurable functions $f(x)$ such that $\int_I dx\,(1+|x|)^q|f(x)|^p<+\infty$, and let $L^p(I)=L^p_0(I)$. Throughout the paper we will use $\|f\|_1$ and $\|f\|_{1,1}$ to denote the $L^1(\mathbf{R})$ and $L^1_1(\mathbf{R})$ norms, $\int_{-\infty}^{\infty}dx\,|f(x)|$ and $\int_{-\infty}^{\infty}dx\,[1+|x|]|f(x)|$, respectively. All the results given in this paper are valid if we assume that P and Q are real valued and belong to $L^1_1(\mathbf{R})$. The existence of the first moment is needed for certain results that involve the limit $k\to 0$. Some results, however, will be proved under the weaker conditions $P,Q\in L^1(\mathbf{R})$.

The scattering solutions of (1.1) and (1.2) are those behaving like e^{ikx} or e^{-ikx} as $x\to\pm\infty$, and such solutions occur when $k^2>0$. Among the scattering solutions are the Jost solution from the left $f_l^{\pm}(k,x)$ and the Jost solution from the right $f_r^{\pm}(k,x)$ satisfying the boundary conditions

$$f_l^{\pm}(k,x)=\begin{cases} e^{ikx}+o(1), & x\to+\infty,\\ \dfrac{1}{T^{\pm}(k)}\, e^{ikx}+\dfrac{L^{\pm}(k)}{T^{\pm}(k)}\, e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{1.3}$$

$$f_r^{\pm}(k,x)=\begin{cases} \dfrac{1}{T^{\pm}(k)}\, e^{-ikx}+\dfrac{R^{\pm}(k)}{T^{\pm}(k)}\, e^{ikx}+o(1), & x\to+\infty,\\ e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{1.4}$$

where $T^{\pm}(k)$ are the transmission coefficients, and $R^{\pm}(k)$ and $L^{\pm}(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrices $\mathbf{S}^{+}(k)$ associated with (1.1) and $\mathbf{S}^{-}(k)$ with (1.2) are given by

$$\mathbf{S}^{\pm}(k)=\begin{bmatrix} T^{\pm}(k) & R^{\pm}(k)\\ L^{\pm}(k) & T^{\pm}(k)\end{bmatrix}.$$

When $P(x)\leqslant 0$, it will be seen that $\mathbf{S}^{+}(k)$ exists for all $k\in\mathbf{R}$; however, when $P(x)\geqslant 0$ or when $P(x)$ has mixed sign, we will see that $\mathbf{S}^{+}(k)$ may not exist at $k=0$ or at some other real values of k.

In this paper we analyze the direct scattering problem in preparation of a more detailed study of various inverse scattering problems for (1.1). One of these inverse problems consists of the recovery of $P(x)$ and $Q(x)$ from an appropriate set of scattering data. In the radial case, when there are no bound states, Jaulent and Jean[1] presented an inversion method with real $Q(x)$ and imaginary $P(x)$. They[2,3] also extended their method to solve the full-line one-dimensional inverse problem for real $Q(x)$ and imaginary $P(x)$. In this method, using the scattering data $\{R^{+}(k),R^{-}(k)\}$, a pair of two coupled Marchenko integral equations is solved and these solutions are used in a first-order ordinary differential equation whose solution leads to $P(x)$. Jaulent[4] also extended this method to the case when $P(x)$ is real although complete details and proofs were not given. When $P(x)$ is purely imaginary and $\int_{-\infty}^{\infty}dz\ P(z)=0$, Sattinger and Szmigielski[5] showed that one can simplify the method of Jaulent and Jean and recover $P(x)$ by solving an algebraic equation rather than a differential equation.

We should also mention the study by Kaup[6] on the direct and inverse scattering problem for

$$\phi''+\left[k^2+\frac{1}{4\beta^2}\right]\phi=[ikP(x)+Q(x)]\phi, \tag{1.5}$$

where β is a nonzero constant and $P,Q\in L_1^1(\mathbf{R})$. Under certain additional assumptions on $P(x)$, Tsutsumi[7] studied the scattering problem for (1.5) with $\beta=\frac{1}{2}$ using a 2×2 matrix analog of (1.1) with k replaced by $\sqrt{k^2+1}$. When P and Q are in the Schwartz space, $\int_{-\infty}^{\infty}dz\ P(z)=0$, and $\beta=\frac{1}{2}$, Sattinger and Szmigielski[8] also studied the inverse scattering problem for (1.5). In Refs. 6 and 8 the inverse scattering problem is analyzed by studying a Riemann–Hilbert problem on a particular Riemann surface.

When $P(x)$ is purely imaginary, the methods available for self-adjoint differential operators can be employed to analyze the inverse scattering problem for (1.1); furthermore, in this case[2] the scattering matrices $\mathbf{S}^{\pm}(k)$ are unitary, and hence the reflection coefficients cannot exceed 1 in absolute value. However, when $P(x)$ is real valued, the differential operator pertaining to (1.1) is no longer self-adjoint and the scattering matrices $\mathbf{S}^{\pm}(k)$ are no longer unitary. Consequently, the analysis of the direct and inverse scattering problems with real $P(x)$ is different and more difficult than with imaginary $P(x)$. The standard proof[9,10] of the absence of singularities of the transmission coefficient for $k\in\mathbf{R}$, which relies heavily on the self-adjointness of the differential operator, breaks down. Since the reflection coefficients may be larger than one in absolute value, the standard proof of the unique solvability of the Marchenko integral equations is no longer valid. Fortunately, when $P(x)\leqslant 0$, some of the usual properties of the one-dimensional Schrödinger equation given in (2.6), such as the simplicity of the poles of the transmission coefficient, the confinement of these poles to the imaginary axis in the upper-half complex plane $\mathbf{C}^+$, and the absence of singularities of the transmission coefficient when $k\in\mathbf{R}$ are still valid for (1.1), and the proofs of such properties are obtained by a variation of the arguments used for (2.6).

This paper is organized as follows. In Secs. II and III, relying on techniques established in a variety of papers,[10–12] we study the analyticity properties of the Jost solutions of (1.1) and analyze their small-k and large-k asymptotics. In Secs. IV–VI we analyze various properties of the scattering matrices $\mathbf{S}^{\pm}(k)$. In Sec. VII we study the change in the scattering coefficients when $P(x)$ and $Q(x)$ are perturbed. In Sec. VIII we analyze the relation between the poles of $T^+(k)$ in $\mathbf{C}^+$ and the bound states; we also study multiple poles of $T^+(k)$ in terms of Jordan chains of the differential operator given in (8.3). Recall that the bound state solutions of (1.1) and (1.2) are those nontrivial solutions belonging to $L^2(\mathbf{R})$. In the radial case when $P(x)\leqslant 0$, under certain additional conditions on $P'(x)$, using the theory of abstract operator polynomials, Pivovarchik has shown that[13] the number of bound states is independent of $P(x)$ and that[14–16] the bound states are simple and can only occur when k is located on the positive imaginary axis. In Sec. IX we study the bound states for (1.1) further and show that the poles of $T^+(k)$ in $\mathbf{C}^+$ can only occur in a certain region in $\mathbf{C}^+$ determined by $P(x)$ and $Q(x)$. When $P(x)\leqslant 0$, we derive Pivovarchik's results in an elementary way without using the theory of abstract operator polynomials and without assuming the differentiability of $P(x)$; we also show that the bound states can only occur at certain negative energies and obtain some lower and upper bounds for these energies. In Sec. IX we also obtain a Levinson theorem relating the number of bound states to the change in the argument of $T^+(k)$, and we show that the number of bound states is unchanged under certain small perturbations of P and Q. In Sec. X we analyze the zeros of the Jost solutions and obtain various results concerning the number and location of these zeros and their relationship to the bound states; we also show that the number of bound states of (1.1) with real energies is greater than or equal to the number of bound states with $P(x)=0$. In Sec. XI, we show by examples that there may be bound states with complex energies and that the multiplicity of a bound state (in the sense of the order to which $1/T^+(k)$ vanishes) may be larger than one. Finally, in the Appendix we obtain various small-k estimates that are needed in the proof of Theorem 5.2.

II. ANALYTICITY AND SMALL-*k* ASYMPTOTICS OF JOST SOLUTIONS

Let us define the Faddeev functions from the left, $m_l^{\pm}(k,x)$, and from the right, $m_r^{\pm}(k,x)$, by

$$m_l^{\pm}(k,x)=e^{-ikx}f_l^{\pm}(k,x), \quad m_r^{\pm}(k,x)=e^{ikx}f_r^{\pm}(k,x).$$

Then $m_l^{\pm}(k,x)$ satisfies

$$m_l^{\pm}(k,x)=1+\frac{1}{2ik}\int_x^{\infty}dy\ [e^{2ik(y-x)}-1][\pm ikP(y)+Q(y)]m_l^{\pm}(k,y), \tag{2.1}$$

$$m_l^{\pm\prime}(k,x)=-\int_x^{\infty}dy\ e^{2ik(y-x)}[\pm ikP(y)+Q(y)]m_l^{\pm}(k,y). \tag{2.2}$$

By $\overline{\mathbf{C}^+}$ we denote $\mathbf{C}^+\cup\mathbf{R}$. In the following theorem and throughout the paper we will use C to denote a generic constant (independent of x and k) that does not necessarily assume the same value at each appearance.

Theorem 2.1: (i) Assume $P,Q\in L^1(\mathbf{R})$. Then, for each $x\in\mathbf{R}$, the functions $m_l^{\pm}(k,x)$, $m_r^{\pm}(k,x)$, $m_l^{\pm\prime}(k,x)$, and $m_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. Consequently, for each $x\in\mathbf{R}$ the Jost solutions $f_l^{\pm}(k,x),f_r^{\pm}(k,x)$ and their derivatives $f_l^{\pm\prime}(k,x),f_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. Moreover, for each $k\in\overline{\mathbf{C}^+}\setminus\{0\}$ we have

$$|m_l^{\pm}(k,x)|\leqslant Ce^{C/|k|},\quad |m_r^{\pm}(k,x)|\leqslant Ce^{C/|k|}, \tag{2.3}$$

$$m_l^{\pm}(k,x)=1+\mathrm{o}(1),\quad m_l^{\pm\prime}(k,x)=\mathrm{o}(1),\quad x\to+\infty,$$

$$m_r^{\pm}(k,x)=1+\mathrm{o}(1),\quad m_r^{\pm\prime}(k,x)=\mathrm{o}(1),\quad x\to-\infty.$$

(ii) Assume $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$. Then the continuity of the functions in (i) extends to $\overline{\mathbf{C}^+}$. Moreover, for $k\in\overline{\mathbf{C}^+}$ we have

$$|m_l^{\pm}(k,x)|\leqslant C[1+\max\{0,-x\}],\quad |m_r^{\pm}(k,x)|\leqslant C[1+\max\{0,x\}], \tag{2.4}$$

$$m_l^{\pm\prime}(k,x)=\mathrm{o}(1/x),\quad x\to+\infty;\quad m_r^{\pm\prime}(k,x)=\mathrm{o}(1/x),\quad x\to-\infty.$$

Proof: The proof is obtained from (2.1), (2.2), and similar equations for $m_r^{\pm}(k,x)$ in a manner analogous[10] to the case with $P(x)=0$. In the proof of (ii), one also uses the estimates

$$|1-e^{2ik(y-x)}|\leqslant 2,\quad |1-e^{2ik(y-x)}|\leqslant 2|k|(y-x),$$

for $k\in\overline{\mathbf{C}^+}$ and $y\geqslant x$. ■

When $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$, using (2.1), (2.2), and Theorem 2.1 (ii), we obtain

$$m_l^{\pm}(k,x)=m_l^{\pm}(0,x)+\mathrm{o}(1),\quad m_l^{\pm\prime}(k,x)=m_l^{\pm\prime}(0,x)+\mathrm{o}(1),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^+}, \tag{2.5}$$

uniformly on compact x-intervals. Let us consider the Schrödinger equation obtained from (1.1) and (1.2) by setting $P(x)=0$, namely

$$\psi^{[0]\prime\prime}(k,x)+k^2\psi^{[0]}(k,x)=Q(x)\psi^{[0]}(k,x),\quad x\in\mathbf{R}. \tag{2.6}$$

Let $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ denote the Jost solutions of (2.6) from the left and from the right, respectively. From (1.1) and the corresponding boundary conditions we see that $f_l^{\pm}(0,x), f_r^{\pm}(0,x)$, and their derivatives are determined by $Q(x)$ alone and

$$m_l^{\pm}(0,x)=f_l^{\pm}(0,x)=f_l^{[0]}(0,x),\quad m_l^{\pm\prime}(0,x)=f_l^{\pm\prime}(0,x)=f_l^{[0]\prime}(0,x), \tag{2.7}$$

As seen from (2.6) and (2.7) we have

$$Q(x)=\frac{f_l^{\pm\prime\prime}(0,x)}{f_l^{\pm}(0,x)}=\frac{f_l^{[0]\prime\prime}(0,x)}{f_l^{[0]}(0,x)}.$$

Let $\mathbf{S}^{[0]}(k)$ denote the scattering matrix associated with (2.6):

$$\mathbf{S}^{[0]}(k)=\begin{bmatrix}T^{[0]}(k) & R^{[0]}(k)\\ L^{[0]}(k) & T^{[0]}(k)\end{bmatrix},$$

where $T^{[0]}(k)$ is the transmission coefficient and $R^{[0]}(k)$ and $L^{[0]}(k)$ are the reflection coefficients from the right and from the left, respectively. Generically $f_l^{\pm}(0,x)$ and $f_r^{\pm}(0,x)$ are linearly independent, but in the so-called exceptional case these two functions are linearly dependent. We have[9,10]

$$[f_l^{[0]}(0,x);f_r^{[0]}(0,x)]=\int_{-\infty}^{\infty}dy\,Q(y)f_l^{[0]}(0,y)=\int_{-\infty}^{\infty}dy\ Q(y)f_r^{[0]}(0,y)=\lim_{k\to0}\frac{-2ik}{T^{[0]}(k)}, \tag{2.8}$$

where $[f;g]=fg'-f'g$ denotes the Wronskian. Thus $T^{[0]}(0)=0$ generically and $T^{[0]}(0)\neq0$ in the exceptional case. In the exceptional case, let us define

$$\gamma=\frac{f_l^{[0]}(0,x)}{f_r^{[0]}(0,x)}. \tag{2.9}$$

Then γ is a nonzero real constant determined by $Q(x)$ alone, and we have $\gamma=f_l^{[0]}(0,-\infty)$.

III. LARGE-*k* ASYMPTOTICS OF JOST SOLUTIONS

In this section we analyze the large-k asymptotics of the Jost solutions. We assume that $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$. The results given here will be used in Sec. VI to obtain the large-k asymptotics of the scattering matrix $\mathbf{S}^+(k)$.

Let us define

$$\eta_l^\pm(k,x)=e^{\pm\zeta}m_l^\pm(k,x)=e^{-ikx\pm\zeta}f_l^\pm(k,x),$$
$$\eta_r^\pm(k,x)=e^{\pm p\mp\zeta}m_r^\pm(k,x)=e^{ikx\pm p\mp\zeta}f_r^\pm(k,x), \tag{3.1}$$

where

$$\zeta=\zeta(x)=\frac{1}{2}\int_x^\infty dz\ P(z),\quad p=\frac{1}{2}\int_{-\infty}^\infty dz\ P(z), \tag{3.2}$$

so that $\int_{-\infty}^x dz\ P(z)/2=p-\zeta$. Thus

$$f_l^\pm(k,x)=e^{ikx\mp\zeta}\eta_l^\pm(k,x),$$
$$f_l^{\pm\prime}(k,x)=e^{ikx\mp\zeta}[(ik\pm P/2)\eta_l^\pm(k,x)+\eta_l^{\pm\prime}(k,x)], \tag{3.3}$$

$$f_r^\pm(k,x)=e^{-ikx\mp p\pm\zeta}\eta_r^\pm(k,x),$$
$$f_r^{\pm\prime}(k,x)=e^{-ikx\mp p\pm\zeta}[(-ik\mp P/2)\eta_r^\pm(k,x)+\eta_r^{\pm\prime}(k,x)]. \tag{3.4}$$

Theorem 3.1: Assume $P,Q\in L^1(\mathbf{R})$. Then, for each $x\in\mathbf{R}$, the functions $\eta_l^\pm(k,x)$ and $\eta_r^\pm(k,x)$ are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$, and we have

$$|\eta_l^\pm(k,x)|\le Ce^{C/|k|},\quad |\eta_r^\pm(k,x)|\le Ce^{C/|k|},\quad k\in\overline{\mathbf{C}^+}\setminus\{0\}. \tag{3.5}$$

If $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$, then, for each $x\in\mathbf{R}$, the functions $\eta_l^\pm(k,x)$ and $\eta_r^\pm(k,x)$ are continuous in $\overline{\mathbf{C}^+}$, and we have

$$|\eta_l^\pm(k,x)|\le C[1+\max\{0,-x\}],\quad |\eta_r^\pm(k,x)|\le C[1+\max\{0,x\}],\quad k\in\overline{\mathbf{C}^+}. \tag{3.6}$$

Moreover, if $P,Q\in L^1(\mathbf{R})$, then

$$\eta_l^\pm(k,x)=1+o(1),\quad \eta_r^\pm(k,x)=1+o(1),\quad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{3.7}$$

$$\eta_l^{\pm\prime}(k,x)=o(k),\quad \eta_r^{\pm\prime}(k,x)=o(k),\quad k\to\infty \text{ in } \overline{\mathbf{C}^+}. \tag{3.8}$$

Proof: The analyticity in $\mathbf{C}^+$, the continuity in $\overline{\mathbf{C}^+}\setminus\{0\}$, and (3.5) and (3.6) follow from (3.1) and Theorem 2.1 (i). To prove (3.7) we use (3.1) in (2.1). Letting $z(k,x)=\eta_l^+(k,x)-1$, after some simplifications, we obtain

$$z(k,x)=z_0(k,x)+\frac{1}{2ik}\int_x^\infty dy\ [e^{2ik(y-x)}-1]e^{\zeta(x)-\zeta(y)}[ikP(y)+Q(y)]z(k,y), \tag{3.9}$$

with

$$z_0(k,x)=\frac{1}{2}\int_x^\infty dy\ e^{2ik(y-x)}e^{\zeta(x)-\zeta(y)}P(y)+\frac{1}{2ik}\int_x^\infty dy\ [e^{2ik(y-x)}-1]e^{\zeta(x)-\zeta(y)}Q(y). \tag{3.10}$$

Using (2.4) in (3.10) we get

$$|z_0(k,x)|\leqslant C_1\int_x^\infty dy\ [|P(y)|+|Q(y)|/|k|],\quad k\in\overline{\mathbf{C}^+}\setminus\{0\},$$

where C_1 is a suitable constant. For each $x\in\mathbf{R}$, let $s_x(k)=\sup_{t\geqslant x}|z_0(k,t)|$. Solving (3.9) by iteration we obtain

$$|z(k,x)|\leqslant s_x(k)e^{C_1\int_x^\infty dy\ [|P(y)|+|Q(y)|/|k|]}. \tag{3.11}$$

Applying the Riemann–Lebesgue lemma to (3.10), we get

$$s_x(k)=\mathrm{o}(1),\quad k\to\pm\infty. \tag{3.12}$$

Using (3.11) and (3.12) we see that $z(k,x)$ is uniformly bounded in $\overline{\mathbf{C}^+}$ for $|k|\geqslant a>0$ for each $x\in\mathbf{R}$ and $a>0$. Hence, in view of (3.12) and a Phragmén–Lindelöf theorem[17] we conclude that $z(k,x)\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. The proof of (3.7) for $\eta_l^-(k,x)$ and $\eta_r^\pm(k,x)$ is similar. To prove (3.8) for $\eta_l^{+\prime}(k,x)$ we introduce the functions

$$\xi_l^\pm(k,x)=\frac{1}{2ik}[\pm P(x)\eta_l^\pm(k,x)+2\eta_l^{\pm\prime}(k,x)],$$
$$\xi_r^\pm(k,x)=\frac{1}{2ik}[\mp P(x)\eta_r^\pm(k,x)+2\eta_r^{\pm\prime}(k,x)]. \tag{3.13}$$

Since $\xi_l^\pm(k,x)=(1/ik)\ m_l^{\pm\prime}(k,x)e^{\pm\zeta}$, it follows from (3.1) and (3.13) that

$$\xi_l^+(k,x)=-\int_x^\infty dy\ [P(y)-iQ(y)/k]e^{2ik(y-x)}e^{\zeta(x)-\zeta(y)}\eta_l^+(k,y). \tag{3.14}$$

Thus, using (3.5), we see that the integrand on the right-hand side of (3.14) is bounded by the integrable function $C_a[|P(y)|+|Q(y)|/a]$, uniformly in x and $k\in\overline{\mathbf{C}^+}$ for $|k|\geqslant a>0$ and each $a>0$, where the constant C_a only depends on a but not on x and k. By a variant of the Riemann–Lebesgue lemma, we conclude that the right-hand side of (3.14) is o(1) as $k\to\pm\infty$, so that by a Phragmén–Lindelöf theorem,[17] we see that the left-hand side of (3.14) is o(1) as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Consequently, $\xi_l^+(k,x)=\mathrm{o}(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$, and so (3.8) for $\eta_l^{+\prime}(k,x)$ follows by using (3.13). The proof of (3.8) for $\eta_l^{-\prime}(k,x)$ and $\eta_r^{\pm\prime}(k,x)$ is similar. ■

Let us also mention that it is possible to study the large k-behavior of the solutions of (1.1) by converting it into a system of two coupled, first-order differential equations. We will not give the details here but refer the interested reader to Ref. 18.

IV. SCATTERING COEFFICIENTS

In this section we summarize some basic facts about the scattering coefficients. In particular, we are concerned with symmetries, Wronskian relations, analyticity, and continuity properties. Applications and refinements of these results will be given in the subsequent sections. Note that, since $P(x)$ is assumed to be real, $\mathbf{S}^\pm(k)$ is not unitary. However, there are certain relations (Proposition 4.3) involving the scattering coefficients which reduce to the usual unitarity relations when $P(x)=0$.

We begin with the observation that when $k\in\mathbf{R}$, the quantities $f_l^-(-k,x)$ and $f_r^-(-k,x)$ are also solutions of (1.1) and hence can be expressed as linear combinations of $f_l^+(k,x)$ and $f_r^+(k,x)$, unless the latter functions are linearly dependent. Using (1.3) and (1.4) we obtain the two vector equations

$$\begin{bmatrix} f_l^\mp(-k,x)\\ f_r^\mp(-k,x)\end{bmatrix}=\begin{bmatrix} T^\pm(k) & -R^\pm(k)\\ -L^\pm(k) & T^\pm(k)\end{bmatrix}\begin{bmatrix} f_r^\pm(k,x)\\ f_l^\pm(k,x)\end{bmatrix},\quad k\in\mathbf{R}. \tag{4.1}$$

In general, $f_l^\mp(-k,x)$ and $f_r^\mp(-k,x)$ cannot be continued to $\mathbf{C}^+$ as functions of k because $f_l^\pm(k,x)$ and $f_r^\pm(k,x)$ usually cannot be extended to the lower-half complex plane $\mathbf{C}^-$.

Using (1.3) and (1.4), the scattering coefficients can be expressed[2] in terms of Wronskians:

$$[f_l^{\pm}(k,x);f_r^{\pm}(k,x)]=-\frac{2ik}{T^{\pm}(k)},\quad k\in\overline{\mathbf{C}^+},\tag{4.2}$$

$$[f_l^{\pm}(k,x);f_r^{\mp}(-k,x)]=\frac{2ikL^{\pm}(k)}{T^{\pm}(k)}=-\frac{2ikR^{\mp}(-k)}{T^{\mp}(-k)},\tag{4.3}$$

$$[f_r^{\pm}(k,x);f_l^{\mp}(-k,x)]=-\frac{2ikR^{\pm}(k)}{T^{\pm}(k)}=\frac{2ikL^{\mp}(-k)}{T^{\mp}(-k)},\tag{4.4}$$

$$[f_l^{\pm}(k,x);f_l^{\mp}(-k,x)]=-2ik=-2ik\,\frac{1-L^{\pm}(k)L^{\mp}(-k)}{T^{\pm}(k)T^{\mp}(-k)},\tag{4.5}$$

$$[f_r^{\pm}(k,x);f_r^{\mp}(-k,x)]=2ik=2ik\,\frac{1-R^{\pm}(k)R^{\mp}(-k)}{T^{\pm}(k)T^{\mp}(-k)}.\tag{4.6}$$

Let an overline denote complex conjugation. It is already known[2] that

$$f_l^{\pm}(-\overline{k},x)=\overline{f_l^{\pm}(k,x)},\quad f_r^{\pm}(-\overline{k},x)=\overline{f_r^{\pm}(k,x)},\quad k\in\overline{\mathbf{C}^+},\tag{4.7}$$

$$f_l^{\pm}(-k,x)=\overline{f_l^{\pm}(k,x)},\quad f_r^{\pm}(-k,x)=\overline{f_r^{\pm}(k,x)},\quad k\in\mathbf{R},\tag{4.8}$$

$$\frac{1}{T^{\pm}(-\overline{k})}=\frac{1}{\overline{T^{\pm}(k)}},\quad k\in\overline{\mathbf{C}^+}\setminus\{0\},\tag{4.9}$$

$$\mathbf{S}^{\pm}(-k)=\overline{\mathbf{S}^{\pm}(k)},\quad \mathbf{S}^{\pm}(k)\mathbf{S}^{\mp}(-k)^t=\mathbf{I},\quad k\in\mathbf{R},\tag{4.10}$$

where $\mathbf{I}$ is the 2×2 unit matrix and the superscript t denotes the matrix transpose. It is understood that (4.10) holds only at the points where the scattering coefficients are defined; we will see later that $\mathbf{S}^{\pm}(k)$ may not be defined for certain real values of k.

From (4.10) we see that

$$T^{\mp}(k)=\frac{T^{\pm}(-k)}{T^{\pm}(-k)^2-L^{\pm}(-k)R^{\pm}(-k)},$$

and therefore

$$\det\,\mathbf{S}^{\pm}(k)=T^{\pm}(k)^2-L^{\pm}(k)R^{\pm}(k)=\frac{T^{\pm}(k)}{T^{\mp}(-k)},\quad k\in\mathbf{R}.\tag{4.11}$$

It follows from (4.9) that the zeros of $1/T^{\pm}(k)$ either lie on or are symmetrically located with respect to the imaginary axis in $\overline{\mathbf{C}^+}$; in particular, $1/T^+(k_0)=0$ for some $k_0\in\mathbf{R}\setminus\{0\}$ implies that $1/T^+(-k_0)=0$.

Proposition 4.1: Assume $P,Q\in L^1(\mathbf{R})$. Then: (i) $1/T^{\pm}(k)$ are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$, and bounded in the sector $\{k\in\overline{\mathbf{C}^+}:|k|\geqslant a>0\}$ for every $a>0$. If, in addition, $Q\in L^1_1(\mathbf{R})$, then $k/[(k+i)T^{\pm}(k)]$ are continuous and bounded in $\overline{\mathbf{C}^+}$, and $kL^{\pm}(k)/T^{\pm}(k)$ and $kR^{\pm}(k)/T^{\pm}(k)$ are continuous and bounded in $\mathbf{R}$. Consequently, $T^{\pm}(k)$ cannot have any zeros in $\overline{\mathbf{C}^+}\setminus\{0\}$. (ii) The zeros of $k/[(k+i)T^{\pm}(k)]$ in $\mathbf{C}^+$ are all isolated, and their accumulation points, if any, must lie on the real axis or at infinity.

Proof: All the assertions follow from (4.3), (4.4), (4.9), and Theorem 2.1. ■

Note that contrary to the case where $P(x)$ is either zero or purely imaginary, we cannot *a priori* rule out possible singularities of the scattering coefficients $T^{\pm}(k)$, $R^{\pm}(k)$, and $L^{\pm}(k)$ on the real axis. In Examples 11.2 and 11.4, we show that $1/T^+(k)$ may have zeros on $\mathbf{R}$ or off the imaginary axis in $\mathbf{C}^+$.

Proposition 4.2: Assume $P,Q\in L^1(\mathbf{R})$. Then, for any $k_0\in\mathbf{R}\setminus\{0\}$, the quantities $1/T^+(k_0)$ and $R^{\pm}(k_0)/T^{\pm}(k_0)$ cannot be zero simultaneously; similarly, $1/T^+(k_0)$ and $L^{\pm}(k_0)/T^{\pm}(k_0)$ cannot be zero simultaneously. If $1/T^+(k_0)=0$ for some $k_0\in\mathbf{R}\setminus\{0\}$, then none of the eight quantities $R^{\pm}(k_0)/T^{\pm}(k_0)$, $L^{\pm}(k_0)/T^{\pm}(k_0)$, $R^{\pm}(-k_0)/T^{\pm}(-k_0)$, $L^{\pm}(-k_0)/T^{\pm}(-k_0)$ are zero. Moreover, $R^+(k)$ is continuous for $k\in\mathbf{R}\setminus\{0\}$ if and only if $1/T^+(k)$ does not vanish for $k\in\mathbf{R}\setminus\{0\}$. Similarly, $L^+(k)$ is continuous for $k\in\mathbf{R}\setminus\{0\}$ if and only if $1/T^+(k)$ does not vanish for $k\in\mathbf{R}\setminus\{0\}$.

Proof: By Proposition 4.1 we know that $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$, $L^{\pm}(k)/T^{\pm}(k)$ are continuous when $k\in\mathbf{R}\setminus\{0\}$. Moreover, the right-hand sides in (4.5) and (4.6) cannot be zero when $k\neq 0$. ■

Note that, if $1/T^+(k_0)=0$ for some $k_0\in\mathbf{R}\setminus\{0\}$, from (4.2) we see that $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ are linearly dependent, and from (1.3) and (1.4) we obtain

$$\frac{f_l^+(k_0,x)}{f_r^+(k_0,x)}=\frac{L^+(k_0)}{T^+(k_0)}=\frac{T^+(k_0)}{R^+(k_0)}.$$

Moreover, if $1/T^+(k)=c(k-k_0)^m+o(k-k_0)^m$ as $k\to k_0$, for some nonzero constant c and positive integer m, then $R(k)=d(k-k_0)^{-m}+o(k-k_0)^{-m}$ as $k\to k_0$ with $d\neq 0$. Thus, we see that $T^+(k)$ is continuous for $k\in\mathbf{R}\setminus\{0\}$ if and only if $R^+(k)$ [or $L^+(k)$] is continuous.

Proposition 4.3: Assume $P,Q\in L^1(\mathbf{R})$. The scattering coefficients satisfy

$$\frac{1}{|T^{\pm}(k)|^2}=1+\left|\frac{L^{\pm}(k)}{T^{\pm}(k)}\right|^2\mp\int_{-\infty}^{\infty}dx\;|f_l^{\pm}(k,x)|^2P(x),\quad k\in\mathbf{R}\setminus\{0\},\tag{4.12}$$

$$\frac{1}{|T^{\pm}(k)|^2}=1+\left|\frac{R^{\pm}(k)}{T^{\pm}(k)}\right|^2\mp\int_{-\infty}^{\infty}dx\;|f_r^{\pm}(k,x)|^2P(x),\quad k\in\mathbf{R}\setminus\{0\}.\tag{4.13}$$

Hence, if $P(x)\leqslant 0$, then $1/T^+(k)$ cannot have any zeros for $k\in\mathbf{R}$, and we have

$$|T^+(k)|^2+|L^+(k)|^2\leqslant 1,\quad |T^+(k)|^2+|R^+(k)|^2\leqslant 1,\quad k\in\mathbf{R}\setminus\{0\}.\tag{4.14}$$

If $1/T^+(k)$ does not have any zeros for $k\in\mathbf{R}\setminus\{0\}$ and $P(x)\geqslant 0$, then we have

$$|T^+(k)|^2+|L^+(k)|^2\geqslant 1,\quad |T^+(k)|^2+|R^+(k)|^2\geqslant 1,\quad k\in\mathbf{R}\setminus\{0\}.\tag{4.15}$$

Moreover, if $P(x)\leqslant 0$, then $1/|T^+(k)|\geqslant 1/|T^-(k)|$ for $k\in\mathbf{R}\setminus\{0\}$.

Proof: From (1.1) and (1.2) we obtain

$$\frac{d}{dx}\,[f_l^{\pm}(-k,x);f_l^{\pm}(k,x)]=\pm 2ikf_l^{\pm}(-k,x)f_l^{\pm}(k,x)P(x),\tag{4.16}$$

$$\frac{d}{dx}\,[f_r^{\pm}(-k,x);f_r^{\pm}(k,x)]=\pm 2ikf_r^{\pm}(-k,x)f_r^{\pm}(k,x)P(x).\tag{4.17}$$

Hence, using (1.3), (1.4), (4.8), (4.10) in (4.16) and (4.17), for $k\in\mathbf{R}\setminus\{0\}$, we obtain (4.12) and (4.13), which imply (4.14) and (4.15). The last inequality follows from (4.10) by subtracting (4.12) from (4.13). ■

V. SMALL-*k* ANALYSIS OF SCATTERING COEFFICIENTS

In this section we analyze the small-k asymptotics of $\mathbf{S}^{\pm}(k)$. Our results will depend on whether we are in the generic or the exceptional case.

Proposition 5.1: Assume $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$ and suppose that we are in the generic case. Then $R^{\pm}(0)=L^{\pm}(0)=-1$, $T^{\pm}(k)$ vanish linearly as $k\to 0$ in $\overline{\mathbf{C}^+}$, and

$$\lim_{k\to 0}\frac{2ik}{T^+(k)}=\lim_{k\to 0}\frac{2ik}{T^-(k)}=\lim_{k\to 0}\frac{2ik}{T^{[0]}(k)}.\tag{5.1}$$

Furthermore, det $\mathbf{S}^{\pm}(0)=-1$, and

$$T^{\pm}(k)=\frac{-2ik}{\int_{-\infty}^{\infty}dy\ Q(y)f_l^{[0]}(0,y)}+\mathrm{o}(k),\quad k\to 0\ \text{ in }\ \overline{\mathbf{C}^{+}}. \tag{5.2}$$

Proof: (5.1) follows from (4.2) and Theorem 2.1 (ii). Since generically $R^{[0]}(0)=L^{[0]}(0)=-1$, from (4.3) and (4.4) in the limit $k\to 0$ we get $R^{\pm}(0)=L^{\pm}(0)=-1$. Then, using (4.11) we obtain det $S^{\pm}(0)=-1$. From (2.8) and (5.1) we obtain (5.2). ■

Under stronger conditions on P and Q, the small-k asymptotics of the scattering coefficients in the generic case were known [see, e.g., Ref. 2 in the case with imaginary $P(x)$]. Next we study the small-k asymptotics of the scattering coefficients in the exceptional case, and we show that $S^{\pm}(0)$ is affected by $P(x)$.

Theorem 5.2: In the exceptional case, under the assumptions $P,Q\in L_1^1(\mathbf{R})$, we have

$$T^{\pm}(0)=\frac{2\gamma}{\gamma^2+1\mp\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,x)^2}, \tag{5.3}$$

$$L^{\pm}(0)=\frac{\gamma^2-1\pm\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,\ x)^2}{\gamma^2+1\mp\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,x)^2}, \tag{5.4}$$

$$R^{\pm}(0)=\frac{1-\gamma^2\pm\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,x)^2}{\gamma^2+1\mp\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,x)^2}, \tag{5.5}$$

where γ is the constant defined in (2.9).

Proof: The technical details of the proof are given in the Appendix. As in the case $P(x)=0$ the difficulty is to prove that the transmission coefficients are continuous at $k=0$. This is only straightforward if $P,Q\in L_2^1(\mathbf{R})$ as[10] in the case $P(x)=0$, but not if $P,Q\in L_1^1(\mathbf{R})$. We obtain $T^{+}(0)$ in (5.3) by using (A18) and (A19) in (A17) and also using (4.2). The value of $T^{-}(0)$ in (5.3) is obtained by changing the sign of $P(x)$. In order to obtain $L^{+}(0)$ in (5.4), as in the displayed equation following (A32) of Ref. 12, we first derive

$$\begin{aligned} f_l(0,0)[f_l^{+}(k,x);f_r^{-}(-k,x)] &= f_r^{-}(-k,0)\left[-ikf_l(0,0)+f_l'(0,0)+\int_0^{\infty}dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right] \\ &\quad -f_l^{+}(k,0)\left[-ikf_l(0,0)+f_l'(0,0)-\int_{-\infty}^{0}dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right], \end{aligned} \tag{5.6}$$

where $\tilde{\psi}(k,x)$ is the function in (A2). Estimating various terms in (5.6) as in the proof of Proposition A.4 in connection with (A17), and also by using (4.3), we obtain $L^{+}(0)$ in (5.4). Similarly we obtain $R^{+}(0)$ in (5.5). The values of $L^{-}(0)$ and $R^{-}(0)$ are obtained from $L^{+}(0)$ and $R^{+}(0)$ by changing the sign of $P(x)$. ■

In the exceptional case, it is possible that $\mathbf{S}^{\pm}(k)$ is discontinuous at $k=0$, and as seen from (5.3)–(5.5) this happens if and only if $\int_{-\infty}^{\infty}dx\ P(x)f_l^{[0]}(0,x)^2=\pm(\gamma^2+1)$. For example, if $Q(x)=0$, we have $f_l^{[0]}(0,x)=f_r^{[0]}(0,x)=1$, and hence $\gamma=1$. Thus

$$\frac{1}{T^{\pm}(0)}=1\mp\frac{1}{2}\int_{-\infty}^{\infty}dx\ P(x). \tag{5.7}$$

Hence, if $\int_{-\infty}^{\infty}dx\ P(x)=\pm 2$, then $\mathbf{S}^{\pm}(0)$ is undefined.

We remark that in the exceptional case, when $\mathbf{S}^{\pm}(k)$ is continuous at $k=0$, we can obtain (5.3)–(5.5) also as follows. In (4.1) let $k\to 0$ and use (2.7) and (2.9) to get

$$\gamma=\frac{f_l^{\pm}(0,x)}{f_r^{\pm}(0,x)}=\frac{T^{\pm}(0)}{1+R^{\pm}(0)}=\frac{T^{[0]}(0)}{1+R^{[0]}0}=\frac{1+L^{[0]}(0)}{T^{[0]}(0)}=\frac{1+L^{\pm}(0)}{T^{\pm}(0)}. \tag{5.8}$$

Combining (5.8) with (4.12) for $k=0$ and eliminating $L^{\pm}(0)/T^{\pm}(0)$, we get

$$1+\gamma^2-\frac{2\gamma}{T^{\pm}(0)}=\pm\int_{-\infty}^{\infty}dx\; P(x)f_l^{[0]}(0,x)^2,$$

which gives us (5.3). Then, using (5.3) in (5.8) we obtain (5.4) and (5.5).

In the exceptional case, since $\mathbf{S}^{\pm}(0)$ is real valued, from (5.8) it is seen that $\mathbf{S}^{\pm}(0)$ is a unitary matrix if and only if $R^{\pm}(0)=-L^{\pm}(0)$. By (5.4) and (5.5), this occurs when the integral $\int_{-\infty}^{\infty}dx\; P(x)f_l^{[0]}(0,x)^2$ vanishes in which case we have $\mathbf{S}^{\pm}(0)=\mathbf{S}^{[0]}(0)$ and det $\mathbf{S}^{\pm}(0)=1$.

From the discussion above and Theorem 5.2 we obtain:

Proposition 5.3: Assume that $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$ in the generic case and that $P,Q\in L_1^1(\mathbf{R})$ in the exceptional case. If any one of the quantities $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$,$L^{\pm}(k)/T^{\pm}(k)$ is discontinuous at $k=0$, then all six are discontinuous at $k=0$ and we are in the generic case. If any one of the quantities $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$, $L^{\pm}(k)/T^{\pm}(k)$ is continuous at $k=0$, then all six are continuous at $k=0$ and we are in the exceptional case. Moreover, the three quantities $T^{+}(k)$, $R^{+}(k)$, $L^{+}(k)$ [or $T^{-}(k)$, $R^{-}(k)$, $L^{-}(k)$] are all continuous on $\mathbf{R}$, or they are all discontinuous on $\mathbf{R}$.

VI. LARGE-*k* ANALYSIS OF SCATTERING COEFFICIENTS

In this section we analyze the large-k asymptotics of $\mathbf{S}^{\pm}(k)$. Similar results were obtained under stronger conditions on $P(x)$ and $Q(x)$ [see, e.g., Ref. 2 in the case with imaginary $P(x)$].

Theorem 6.1: Assume $P,Q\in L^1(\mathbf{R})$. Then

$$\frac{1}{T^{\pm}(k)}\,e^{\pm p}=1+o(1),\quad k\to\infty \text{ in } \overline{\mathbf{C}^{+}}, \tag{6.1}$$

$$\frac{R^{\pm}(k)}{T^{\pm}(k)}=o(1),\quad \frac{L^{\pm}(k)}{T^{\pm}(k)}=o(1),\quad k\to\pm\infty, \tag{6.2}$$

where p is the constant defined in (3.2).

Proof: From (4.2) using (3.3) and (3.4), we obtain

$$\frac{2ik}{T^{\pm}(k)}\,e^{\pm p}=[2ik\pm P(x)]\,\eta_l^{\pm}(k,x)\,\eta_r^{\pm}(k,x)+\eta_l^{\pm\prime}(k,x)\,\eta_r^{\pm}(k,x)-\eta_l^{\pm}(k,x)\,\eta_r^{\pm\prime}(k,x). \tag{6.3}$$

Now (6.1) follows from (3.7), (3.8), and (6.3). Similarly, using (3.3), (3.4), and (4.4), we have

$$-\frac{2ikR^{\pm}(k)}{T^{\pm}(k)}=e^{-2ikx\mp p\pm 2\zeta}[\,\eta_r^{\pm}(k,x);\eta_l^{\mp}(-k,x)],\quad k\in\mathbf{R}, \tag{6.4}$$

and the first relation in (6.2) follows by using (3.7) and (3.8) in (6.4). The proof of the second relation in (6.2) is analogous, using (4.3). ■

Note that from (6.1) it follows that e^p is known when either of $T^{\pm}(k)$ is known. From Proposition 4.1 and Theorem 6.1, we have:

Corollary 6.2: Assume $P,Q\in L^1(\mathbf{R})$. If $1/T^{\pm}(k)$ does not vanish for $k\in\mathbf{R}$, then its number of zeros in $\mathbf{C}^{+}$ is finite.

VII. PERTURBATION OF SCATTERING COEFFICIENTS

In this section we establish the stability of $1/T^+(k)$, $R^+(k)/T^+(k)$, and $L^+(k)/T^+(k)$ under small perturbations of P and Q in the norm of either $L^1(\mathbf{R})$ or $L_1^1(\mathbf{R})$. We will not state the results for $R^+(k)/T^+(k)$, since they are identical to those for $L^+(k)/T^+(k)$. An application of these results will be given in Sec. IX.

Given two sets of potentials $P_j(x)$ and $Q_j(x)$ with $j=1,2$, we consider the generalized Schrödinger equations

$$\psi_j^{+\prime\prime}(k,x)+k^2\psi_j^+(k,x)=[ikP_j(x)+Q_j(x)]\psi_j^+(k,x),\quad x\in\mathbf{R},\tag{7.1}$$

and denote their corresponding Faddeev solutions by $m_{l;j}^+(k,x)$ and $m_{r;j}^+(k,x)$, their transmission coefficients by $T_j^+(k)$, and their reflection coefficients from the right and from the left by $R_j^+(k)$ and $L_j^+(k)$, respectively.

Proposition 7.1: Assume $P_j,Q_j\in L^1(\mathbf{R})$ for $j=1,2$. Then for $k\in\overline{\mathbf{C}^+}$ with $|k|\geqslant 1$, we have

$$\left|\frac{1}{T_1^+(k)}-\frac{1}{T_2^+(k)}\right|\leqslant C(\|P_1-P_2\|_1+\|Q_1-Q_2\|_1),\tag{7.2}$$

and for $k\in\mathbf{R}$ with $|k|\geqslant 1$, we have

$$\left|\frac{L_1^+(k)}{T_1^+(k)}-\frac{L_2^+(k)}{T_2^+(k)}\right|\leqslant C(\|P_1-P_2\|_1+\|Q_1-Q_2\|_1).\tag{7.3}$$

Proof: First, by iterating (2.1) and using (2.3), we obtain

$$|m_{l;1}^+(k,x)-m_{l;2}^+(k,x)|\leqslant C_1(\|P_1-P_2\|_1+\|Q_1-Q_2\|_1),\tag{7.4}$$

for some constant C_1. Furthermore, from (1.3) and (2.1) we obtain

$$\frac{1}{T^\pm(k)}=1-\frac{1}{2ik}\int_{-\infty}^{\infty}dy\ [\pm ikP(y)+Q(y)]m_l^\pm(k,y),\tag{7.5}$$

$$\frac{L^\pm(k)}{T^\pm(k)}=\frac{1}{2ik}\int_{-\infty}^{\infty}dy\ e^{-2iky}[\pm ikP(y)+Q(y)]m_l^\pm(k,y).\tag{7.6}$$

Now using (7.4) in (7.5) we get (7.2). The proof of (7.3) is similarly obtained by using (7.4) and (7.6). ■

Proposition 7.2: Assume $P_j\in L^1(\mathbf{R})$ and $Q_j\in L_1^1(\mathbf{R})$ for $j=1,2$. Then, for $k\in\overline{\mathbf{C}^+}$ with $|k|\leqslant 1$, we have

$$\left|\left(\frac{k}{T_1^+(k)}-\frac{k}{T_2^+(k)}\right)\right|\leqslant C(\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}),\tag{7.7}$$

and for $k\in\mathbf{R}$ with $|k|\leqslant 1$, we have

$$\left|\left(\frac{kL_1^+(k)}{T_1^+(k)}-\frac{kL_2^+(k)}{T_2^+(k)}\right)\right|\leqslant C(\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}).\tag{7.8}$$

Proof: Iterating (2.1) with the help of (2.4) and the inequality

$$\frac{1+\max\{0,-y\}}{1+\max\{0,-x\}}\,(y-x)\leqslant 1+|y|,\quad y\geqslant x,\tag{7.9}$$

we obtain

$$|km_{l;j}^+(k,x)|\leqslant C_1,\quad |k|\leqslant 1\tag{7.10}$$

$$|m_{l;1}^{+}(k,x)-m_{l;2}^{+}(k,x)|\leqslant C_2[1+\max\{0,-x\}](\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}), \tag{7.11}$$

$$|km_{l;1}^{+}(k,x)-km_{l;2}^{+}(k,x)|\leqslant C_3(\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}), \quad |k|\leqslant 1, \tag{7.12}$$

for some constants C_1, C_2, and C_3. Then (7.7) and (7.8) follow from using (2.4), (7.10)–(7.12) in (7.5) and (7.6). ■

Proposition 7.3: Assume $P_j, Q_j \in L_1^1(\mathbf{R})$ for $j=1,2$ and $Q_1(x)=Q_2(x)$. Then, we have

$$\left|\frac{1}{T_1^+(k)}-\frac{1}{T_2^+(k)}\right|\leqslant C\|P_1-P_2\|_{1,1}, \quad k\in\overline{\mathbf{C}^+}, \tag{7.13}$$

$$\left|\frac{L_1^+(k)}{T_1^+(k)}-\frac{L_2^+(k)}{T_2^+(k)}\right|\leqslant C\|P_1-P_2\|_{1,1}, \quad k\in\mathbf{R}. \tag{7.14}$$

Proof: First from (2.4), (7.9), and (2.1) we get

$$|m_{l;1}^{+}(k,x)-m_{l;2}^{+}(k,x)|\leqslant C_1|k|[1+\max\{0,-x\}]\|P_1-P_2\|_{1,1}, \tag{7.15}$$

for some constant C_1. Using (2.4), (7.11), and (7.15) in (7.5) and (7.6) we obtain (7.13) and (7.14). ■

VIII. BOUND STATES AND JORDAN CHAINS

Recall that the bound states of (1.1) are its nontrivial solutions belonging to $L^2(\mathbf{R})$. In this section we show that the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ correspond to the bound states of (1.1). We also analyze the order of each zero of $1/T^+(k)$ in $\mathbf{C}^+$ in terms of Jordan chains of the differential operator $\mathbf{W}(k)$ defined in (8.3).

Proposition 8.1: Assume $P,Q\in L^1(\mathbf{R})$. A point $k_0\in\mathbf{C}^+$ corresponds to a bound state of (1.1) if and only if $1/T^+(k_0)=0$. If $k_0\in\mathbf{R}\setminus\{0\}$ and $1/T^+(k_0)=0$, then k_0 does not correspond to a bound state of (1.1); if we further assume $Q\in L_1^1(\mathbf{R})$, then $k=0$ cannot correspond to a bound state even when $1/T^+(0)=0$.

Proof: The first assertion follows from (4.2) and Theorem 2.1 (i). For $k_0\in\mathbf{R}\setminus\{0\}$ every nontrivial solution of (1.1) has the asymptotic form $c_+e^{ik_0x}+c_-e^{-ik_0x}+o(1)$ as $x\to+\infty$, with the constants c_+ and c_- not both equal to zero, and hence cannot be in $L^2(\mathbf{R})$. If $Q\in L_1^1(\mathbf{R})$, then $k_0=0$ cannot be a bound state because any nontrivial solution of (1.1) for $k=0$ has the asymptotic form $c_1x[1+o(1)]+c_2+o(1)$ as $x\to+\infty$, with c_1 and c_2 not both equal to zero. Hence, the proof is complete. ■

Next we analyze multiple poles of $T^+(k)$. Let us differentiate (1.1) with $\psi=f_l^+(k,x)$ or $\psi=f_r^+(k,x)$ with respect to k repeatedly. Defining

$$g_{l,n}^{+}(k,x)=\frac{1}{n!}\left(\frac{\partial}{\partial k}\right)^n f_l^+(k,x), \quad g_{r,n}^{+}(k,x)=\frac{1}{n!}\left(\frac{\partial}{\partial k}\right)^n f_r^+(k,x), \quad n=0,1,2,..., \tag{8.1}$$

and $g_{l,n}^{+}(k,x)=g_{r,n}^{+}(k,x)=0$ for $n=-1,-2,...,$ we obtain the coupled system of differential equations

$$\begin{aligned} &g_{l,n}^{+\prime\prime}(k,x)+k^2g_{l,n}^{+}(k,x)+2kg_{l,n-1}^{+}(k,x)+g_{l,n-2}^{+}(k,x)\\ &\quad=[ikP(x)+Q(x)]g_{l,n}^{+}(k,x)+iP(x)g_{l,n-1}^{+}(k,x),\\ &g_{r,n}^{+\prime\prime}(k,x)+k^2g_{r,n}^{+}(k,x)+2kg_{r,n-1}^{+}(k,x)+g_{r,n-2}^{+}(k,x)\\ &\quad=[ikP(x)+Q(x)]g_{r,n}^{+}(k,x)+iP(x)g_{r,n-1}^{+}(k,x). \end{aligned} \tag{8.2}$$

Defining the differential operator

$$\mathbf{W}(k)=-k^2-\frac{d^2}{dx^2}+ikP+Q, \tag{8.3}$$

so that $\dot{\mathbf{W}}(k) = -2k + iP$ and $\ddot{\mathbf{W}}(k) = -2\mathbf{I}$, we obtain the system of linear equations

$$\mathbf{T}(k)\mathbf{g}_l^+(k,x) = \mathbf{0}, \tag{8.4}$$

where $\mathbf{0}$ is the zero column vector of m components, $\mathbf{g}_l^+(k,x)$ is the column vector

$$[g_{l,m-1}^+(k,x),\ldots,g_{l,0}^+(k,x)]^t,$$

and $\mathbf{T}(k)$ is the $m\times m$ Toeplitz matrix given by

$$\mathbf{T}(k) = \begin{bmatrix} \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mathbf{W}(k) \end{bmatrix}.$$

Using the Leibnitz formula for repeated derivatives of products, we find from (4.2)

$$\frac{1}{n!}\left(\frac{d}{dk}\right)^n \frac{-2ik}{T^+(k)} = \sum_{j=0}^{n} [g_{l,j}^+(k,x); g_{r;n-j}^+(k,x)]. \tag{8.5}$$

We call $k_0 \in \mathbf{C}^+$ an eigenvalue of $\mathbf{W}(k)$ if there exists a nontrivial $\phi \in L^2(\mathbf{R})$ such that $\mathbf{W}(k_0)\phi = 0$. Because of Proposition 8.1, this is equivalent to $1/T^+(k_0) = 0$. Further, ϕ is called an eigenfunction of $\mathbf{W}(k)$ corresponding to the eigenvalue k_0. More generally,[19] if k_0 is an eigenvalue of $\mathbf{W}(k)$, then the string of functions $\phi_0,\ldots,\phi_{m-1}$ in $L^2(\mathbf{R})$ is called a Jordan chain of length m corresponding to the eigenvalue k_0 if $\phi_0 \neq 0$ and (8.4) holds with $[g_{l,m-1}^+(k_0,x),\ldots,g_{l,0}^+(k_0,x)]^t$ replaced by the column vector $[\phi_{m-1}(k_0,x),\ldots,\phi_0(k_0,x)]^t$.

Proposition 8.2: Assume $P,Q \in L^1(\mathbf{R})$ and let $k_0 \in \mathbf{C}^+$ be an eigenvalue of $\mathbf{W}(k)$. If $\{g_{l,j}^+(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m at the eigenvalue k_0, then for $n=0,1,\ldots,m-1$, we have as $x \to \pm\infty$

$$g_{l,n}^+(k_0,x) = \mathrm{O}[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \quad g_{r,n}^+(k_0,x) = \mathrm{O}[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \tag{8.6}$$

$$g_{l,n}^{+\prime}(k_0,x) = \mathrm{O}[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \quad g_{r,n}^{+\prime}(k_0,x) = \mathrm{O}[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}]. \tag{8.7}$$

Proof: If $k_0 \in \mathbf{C}^+$ is an eigenvalue of $\mathbf{W}(k)$, by Proposition 8.1 we have $1/T^+(k_0)=0$. Hence $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ are linearly dependent, and thus we need to prove (8.6) and (8.7) only for $g_{l,n}^+(k_0,x)$ and $g_{l,n}^{+\prime}(k_0,x)$. For $n=0$ these follow from Theorem 2.1 (i). The following argument used to prove (8.6) and (8.7) for $n=m-1$ can be used recursively for $n=1,\ldots,m-2$. Note that, for each $k \in \mathbf{C}^+$, (1.1) has[20] an unbounded solution $X(k,x)$ such that

$$X(k,x) = \mathrm{O}(e^{|x|\operatorname{Im} k}), \quad X'(k,x) = \mathrm{O}(e^{|x|\operatorname{Im} k}), \quad x \to \pm\infty. \tag{8.8}$$

Let us choose $X(k_0,x)$ such that $[f_l^+(k_0,x); X(k_0,x)] = 1$. Let us consider (8.2) as a second-order linear, nonhomogeneous differential equation for $g_{l,m-1}^+(k_0,x)$ and solve it by variation of parameters using the linearly independent solutions $f_l^+(k_0,x)$ and $X(k_0,x)$ of (1.1). We obtain

$$\begin{aligned} g_{l,m-1}^+(k_0,x) = a_{m-1} f_l^+(k_0,x) + b_{m-1} X(k_0,x) - \int_0^x dy ([iP(y) - 2k_0] g_{l,m-2}^+(k_0,y) \\ - g_{l,m-3}^+(k_0,y))[f_l^+(k_0,x)X(k_0,y) - f_l^+(k_0,y)X(k_0,x)], \end{aligned} \tag{8.9}$$

with arbitrary constants a_{m-1} and b_{m-1}. Since $g_{l,n}(k_0,\cdot)\in L^2(\mathbf{R})$ for $n=0,1,\ldots,\ m-2$ and $X(k_0,x)$ is unbounded as $x\to\pm\infty$, the term proportional to $X(k_0,x)$ in (8.9) must vanish. Thus we must have

$$b_{m-1}+\int_0^\infty dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y)=0,$$

$$b_{m-1}-\int_{-\infty}^\infty dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y)=0,$$

and hence

$$\int_{-\infty}^\infty dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y)=0. \tag{8.10}$$

Using (8.10), we can write (8.9) as

$$g_{l,m-1}^+(k_0,x)=A(k_0,x)X(k_0,x)+f_l^+(k_0,x)\times\left[a_{m-1}-\int_0^x dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))X(k_0,y)\right], \tag{8.11}$$

where we have

$$A(k_0,x)=\begin{cases}\int_{-\infty}^x dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y), & x\leqslant 0,\\ \int_x^\infty dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y), & x\geqslant 0.\end{cases} \tag{8.12}$$

Using (8.6) for $n=0,1,\ldots,\ m-2$, (8.8), and (8.12), we obtain (8.6) for $g_{l,m-1}^+(k_0,x)$. Differentiating (8.11) and using (8.10), we obtain

$$g_{l,m-1}^{+\prime}(k_0,x)=[a_{m-1}-I_1(x)]f_l^{+\prime}(k_0,x)+X'(k_0,x)I_2(x),$$

where

$$I_1(x)=\int_0^x dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))X(k_0,y),$$

$$I_2(x)=\int_{-\infty}^x dy([iP(y)-2k_0]g_{l,m-2}^+(k_0,y)-g_{l,m-3}^+(k_0,y))f_l^+(k_0,y).$$

Finally, we obtain (8.7) for $g_{l,m-1}^{+\prime}(k_0,x)$ by using (8.6) for $n=0,1,\ldots,\ m-2$, (8.8), and (8.12).■

Proposition 8.3: Assume $P,Q\in L^1(\mathbf{R})$ and let $k_0\in\mathbf{C}^+$ be an eigenvalue of $\mathbf{W}(k)$. Then for $n=0,1,\ldots,\ m-1$, we have

$$\int_{-\infty}^\infty dy([iP(y)-2k_0]g_{l,n-1}^+(k_0,y)-g_{l,n-2}^+(k_0,y))f_l^+(k_0,y)=0 \tag{8.13}$$

if and only if $\{g_{l,j}^+(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.

Proof: If (8.13) holds for $n=1$, then we must have $f_l^+(k_0,\cdot)\in L^2(\mathbf{R})$; from Proposition 8.2 and its proof it is seen that $f_l^+(k_0,\cdot)\in L^2(\mathbf{R})$ only if $g_{l,0}^+(k_0,x)=f_l^+(k_0,x)$ is an eigenvector of $\mathbf{W}(k)$. Recursively, we can show that $g_{l,n}^+$ given in (8.1) satisfies (8.4), (8.6), and (8.7), and hence

$g_{l,n}^{+}(k_0,\cdot)\in L^2(\mathbf{R})$ for $n=1,\ldots,m-1$. Thus $\{g_{l,j}^{+}(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0. The converse is proved by proceeding recursively as in the proof of Proposition 8.2 leading to (8.10). ■

Theorem 8.4: Assume $P,Q\in L^1(\mathbf{R})$ and let $k_0\in\mathbf{C}^{+}$. Then the following four statements are equivalent:

(a) $\mathbf{W}(k)$ has a Jordan chain of length m corresponding to the eigenvalue k_0.
(b) $\{g_{l,j}^{+}(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.
(c) $\{g_{r,j}^{+}(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.
(d) $1/T^{+}(k)$ has a zero at k_0 of order at least m.

Proof: Clearly (b) implies (a). Now assume (a) holds and let $\{\phi_j\}_{j=0}^{m-1}$ be a Jordan chain of $\mathbf{W}(k)$ of length m at the eigenvalue k_0. Then ϕ_0 must be proportional to $f_l^{+}(k_0,x)$ and $f_r^{+}(k_0,x)$ because the latter two are linearly dependent and $\phi_0(k_0,x)$ is a solution of (1.1) for $k=k_0$. Thus we have $\mathbf{W}(k_0)g_{l,0}^{+}(k_0,x)=0$ and consequently (8.2) is satisfied for $n=0,1,\ldots,m-1$. Hence (b) holds.

Note that (b) and (c) are equivalent because, if $\mathbf{W}(k_0)g_{l,0}^{+}(k_0,x)=0$ for some $k_0\in\mathbf{C}^{+}$ and $g_{l,0}^{+}(k_0,\cdot)\in L^2(\mathbf{R})$, by Proposition 8.1 we must have $1/T^{+}(k_0)=0$ and hence $g_{r,0}^{+}(k_0,x)$ must be a constant multiple of $g_{l,0}^{+}(k_0,x)$.

If (b) holds, then (8.6) and (8.7) must hold for $n=0,1,\ldots,m-1$ because of Proposition 8.2. Then, for $n=0,1,\ldots,m-1$, by evaluating the right-hand side of (8.5) at $x=-\infty$ or at $x=+\infty$, we find that its left-hand side must be zero and thus (d) holds. Now assume that (d) holds and let us show that (b) is true. By Proposition 8.3 it is sufficient to show that (8.13) is satisfied for $n=0,1,\ldots,m-1$. We will do this recursively. First notice that (8.13) trivially holds for $n=0$ because $g_{l,-1}^{+}(k_0,x)=g_{l,-2}^{+}(k_0,x)=0$ and that (8.6) and (8.7) hold for $g_{l,0}^{+}$ and $g_{l,0}^{+\prime}$, respectively, because $1/T^{+}(k_0)=0$ and thus $f_l^{+}(k_0,x)$ is exponentially decaying as $|x|\to\infty$. For $n=1,\ldots,m-2$, the proofs of (8.13) and of (8.6) and (8.7) for $g_{l,n}^{+}$ and $g_{l,n}^{+\prime}$, respectively, are similar to the case when $n=m-1$. Thus, it suffices to give the proofs for $n=m-1$ by assuming that these equations hold for $n=1,\ldots,m-2$. Using (1.1) for $g_{r,0}^{+}(k_0,x)$ and (8.2) for $g_{l,m-1}^{+}(k_0,x)$, we obtain the Wronskian relation

$$-\frac{d}{dx}[g_{l,m-1}^{+}(k_0,x);g_{r,0}^{+}(k_0,x)]=([iP(x)-2k_0]g_{l,m-2}^{+}(k_0,x)-g_{l,m-3}^{+}(k_0,x))g_{r,0}^{+}(k_0,x). \tag{8.14}$$

In a similar way, we obtain

$$\frac{d}{dx}[g_{l,0}^{+}(k_0,x);g_{r,m-1}^{+}(k_0,x)]=([iP(x)-2k_0]g_{r,m-2}^{+}(k_0,x)-g_{r,m-3}^{+}(k_0,x))g_{l,0}^{+}(k_0,x). \tag{8.15}$$

Integrating (8.14) and (8.15) we get

$$\int_{-\infty}^{\infty}dy\ ([iP(y)-2k_0]g_{l,m-2}^{+}(k_0,y)-g_{l,m-3}^{+}(k_0,y))g_{r,0}^{+}(k_0,y)$$
$$=\lim_{x\to-\infty}[g_{l,m-1}^{+}(k_0,x);g_{r,0}^{+}(k_0,x)]-\lim_{x\to+\infty}[g_{l,m-1}^{+}(k_0,x);g_{r,0}^{+}(k_0,x)], \tag{8.16}$$

$$\int_{-\infty}^{\infty}dy\ ([iP(y)-2k_0]g_{r,m-2}^{+}(k_0,y)-g_{r,m-3}^{+}(k_0,y))g_{l,0}^{+}(k_0,y)$$
$$=\lim_{x\to+\infty}[g_{l,0}^{+}(k_0,x);g_{r,m-1}^{+}(k_0,x)]-\lim_{x\to-\infty}[g_{l,0}^{+}(k_0,x);g_{r,m-1}^{+}(k_0,x)]. \tag{8.17}$$

Because (8.6) and (8.7) hold for $n=0,1,\ldots,m-2$, we have

$$\lim_{x\to\pm\infty}\sum_{j=1}^{m-2}[g^+_{l,j}(k_0,x);g^+_{r;m-j}(k_0,x)]=0. \tag{8.18}$$

Since $1/T^+(k_0)$ is assumed to have a zero of order at least m, using (8.5) for $n=m-1$ and (8.18), we obtain

$$\lim_{x\to\pm\infty}([g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]+[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)])=0. \tag{8.19}$$

Using the linear dependence of $g^+_{l,0}(k_0,x)$ and $g^+_{r,0}(k_0,x)$ and the exponential decay of $g^+_{l,m-1}(k_0,x)$ as $x\to+\infty$ and of $g^+_{r,m-1}(k_0,x)$ as $x\to-\infty$, we first conclude that

$$\lim_{x\to+\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]=0,\quad \lim_{x\to-\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]=0. \tag{8.20}$$

Now, from (8.19) and (8.20) we see that

$$\lim_{x\to-\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]=0,\quad \lim_{x\to+\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]=0, \tag{8.21}$$

and thus using (8.20) and (8.21) in (8.16) and (8.17), we get

$$\int_{-\infty}^{\infty}dy\ ([iP(y)-2k_0]g^+_{l,m-1}(k_0,y)-g^+_{l,m-2}(k_0,y))g^+_{r,0}(k_0,y)=0.$$

Thus (8.13) is proved for $n=m-1$, and hence (b) holds. ■

IX. BOUND STATES AND POLES OF $T^+(k)$

In this section we further analyze the poles of $T^+(k)$ in $\mathbf{C}^+$. We show that such poles cannot occur in certain regions in $\mathbf{C}^+$ determined in terms of the constants defined in (9.1). When $P(x)\leqslant 0$, we show that such poles are confined to a certain interval on the positive imaginary axis. We analyze the change in the number of bound states when P and Q are perturbed. In the generic case we find that the number of bound states is unchanged under small perturbations of P and Q; in the exceptional case we find that the number of bound states is unchanged under small perturbations of $P(x)$. When $P(x)\leqslant 0$ we show that the number of bound states is independent of $P(x)$. We also present a Levinson theorem relating the number of bound states to the change in the argument of $T^+(k)$.

Next we obtain some simple conditions on $P(x)$ and $Q(x)$ guaranteeing that there are no bound states outside certain k-regions in $\mathbf{C}^+$ determined by the following parameters:

$$P_{\min}=\operatorname*{ess\,inf}_{x\in\mathbf{R}}P(x),\quad P_{\max}=\operatorname*{ess\,sup}_{x\in\mathbf{R}}P(x),\quad Q_{\min}=\operatorname*{ess\,inf}_{x\in\mathbf{R}}Q(x). \tag{9.1}$$

Let us also define $\beta^*=P_{\max}/2+\sqrt{P^2_{\max}/4-Q_{\min}}$. Note that if $P,Q\in L^1(\mathbf{R})$, then it follows that $P_{\max}\geqslant 0$ with the equality holding if and only if $P(x)\leqslant 0$, that $Q_{\min}\leqslant 0$ with the equality holding if and only if $Q(x)\geqslant 0$, and that $P_{\min}\leqslant 0$ with the equality holding if and only if $P(x)\geqslant 0$. Furthermore, $\beta^*\geqslant P_{\max}$ with the equality holding if and only if $Q(x)\geqslant 0$. Note also that $\beta^*\geqslant 0$ with the equality holding if and only if $P(x)=Q(x)=0$; hence, the case $\beta^*=0$ is trivial.

Theorem 9.1: Assume $P,Q\in L^1(\mathbf{R})$, $P(x)\not\equiv 0$, and $P_{\max}$ is finite. Then for $P_{\max}/2\leqslant \mathrm{Im}\,k<\beta^*$ the zeros of $1/T^+(k)$ can only occur on the imaginary axis, and all such zeros are simple. If, in addition, $Q_{\min}$ is finite, then there are no zeros of $1/T^+(k)$ in the region $\{k\in\mathbf{C}^+:(\mathrm{Im}\,k)^2-(\mathrm{Re}\,k)^2-(\mathrm{Im}\,k)P_{\max}\geqslant -Q_{\min}\}$. Consequently, $1/T^+(k)$ has no zeros in $\mathbf{C}^+$ satisfying $\mathrm{Im}\,k\geqslant\beta^*$.

Proof: From (1.1), after using (4.7), we obtain

$$\frac{d}{dx}\{f^+_l(-\overline{k_0},x)f^{+\prime}_l(k_0,x)\}=|f^{+\prime}_l(k_0,x)|^2+[-k_0^2+ik_0P(x)+Q(x)]|f^+_l(k_0,x)|^2. \tag{9.2}$$

If $k_0 \in \mathbf{C}^+$ is a zero of $1/T^+(k)$, then by integrating (9.2) and using (4.7) and Theorem 2.1 (i), we obtain

$$\int_{-\infty}^{\infty} dx\ |f_l^{+\prime}(k_0,x)|^2 = \int_{-\infty}^{\infty} dx\ [k_0^2 - ik_0 P(x) - Q(x)]|f_l^+(k_0,x)|^2. \tag{9.3}$$

Letting $k_0 = \alpha + i\beta$ and separating the real and imaginary parts in (9.3), we obtain

$$i\alpha \int_{-\infty}^{\infty} dx\ [2\beta - P(x)]|f_l^+(k_0,x)|^2 = 0, \tag{9.4}$$

$$\int_{-\infty}^{\infty} dx\ |f_l^{+\prime}(k_0,x)|^2 = \int_{-\infty}^{\infty} dx\ [\alpha^2 - \beta^2 + \beta P(x) - Q(x)]|f_l^+(k_0,x)|^2. \tag{9.5}$$

From (9.4) we see that we must have $\alpha = 0$ when $P_{\max} \leqslant 2\beta$, and hence any zero of $1/T^+(k)$ with $\mathrm{Im}\, k \geqslant P_{\max}/2$ can only occur on the positive imaginary axis. All such zeros are simple; otherwise, a zero of order two or higher would imply (8.13) with $n=1$, i.e., $\int_{-\infty}^{\infty} dx\ [P(x) - 2\,\mathrm{Im}\, k_0]|f_l^+(k_0,x)|^2 = 0$, which cannot happen if $\mathrm{Im}\, k_0 \geqslant P_{\max}/2$. From (9.5) we see that we cannot have $\alpha^2 - \beta^2 + \beta P(x) - Q(x) \leqslant 0$. Hence there are no zeros of $1/T^+(k)$ in $\{\alpha + i\beta \in \mathbf{C}^+ : \beta^2 - \alpha^2 - \beta P_{\max} \geqslant -Q_{\min}\}$. The analysis of the corresponding region in the $\alpha\beta$-plane indicates that there cannot be any zeros of $1/T^+(k)$ on the imaginary axis when $\mathrm{Im}\, k \geqslant \beta^*$, and hence there cannot be any zeros of $1/T^+(k)$ either on or off the imaginary axis when $\mathrm{Im}\, k \geqslant \beta^*$. ■

When $P(x) \leqslant 0$, from Theorem 9.1 we obtain the following corollary.

Corollary 9.2: Assume $P(x) \leqslant 0$ and $P, Q \in L^1(\mathbf{R})$. Then, the poles of $T^+(k)$ in $\mathbf{C}^+$ are all purely imaginary and simple. In addition, assume that $Q_{\min}$ defined in (9.1) is finite; then there are no zeros of $1/T^+(k)$ in $\mathbf{C}^+$ for $\mathrm{Im}\, k \geqslant \sqrt{-Q_{\min}}$.

Theorem 9.3: Assume $Q(x) \equiv 0$ and $P \in L_1^1(\mathbf{R})$. If $\int_{-\infty}^{\infty} dx\ P(x) > 2$, then (1.1) has at least one bound state at $k = i\beta$ for some positive β. If $\int_{-\infty}^{\infty} dx\ |P(x)| \leqslant 2$, then $1/T^+(k)$ has no zeros in $\mathbf{C}^+$.

Proof: When $\int_{-\infty}^{\infty} dx\ P(x) > 2$, from (5.7) we see that $1/T^+(i\beta)$ is negative at $\beta = 0$ and from (6.1) we see that it is positive as $\beta \to +\infty$. Being a real-valued, continuous function of β, $1/T^+(i\beta)$ must have a zero for some positive β. Now let us prove the second statement. Assume $k \in \mathbf{C}^+$ corresponds to a bound state; we can transform (1.1) into

$$\varphi(k,x) = \int_{-\infty}^{\infty} dy\ \mathscr{B}(k;x,y)\varphi(k,y), \tag{9.6}$$

where we have defined

$$\varphi(k,x) = |P(x)|^{1/2}\psi^+(k,x), \quad \mathscr{B}(k;x,y) = \tfrac{1}{2} e^{ik|x-y|}|P(x)|^{1/2}P(y)/|P(y)|^{1/2}.$$

When $P \in L^1(\mathbf{R})$ and $k \in \mathbf{C}^+$, the integral operator in (9.6) is Hilbert–Schmidt with the Hilbert–Schmidt norm

$$\|\mathscr{B}\|_{\mathrm{HS}}^2 = \tfrac{1}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |P(x)| e^{-2|x-y|\mathrm{Im}\, k}|P(y)|,$$

and hence, if $k \in \mathbf{C}^+$ and $\int_{-\infty}^{\infty} dx\ |P(x)| \leqslant 2$, we have $\|\mathscr{B}\|_{\mathrm{HS}} < 1$. Thus the operator norm of that integral operator is also strictly less than 1 and hence $\varphi = 0$, implying that there cannot be any bound states of (1.1) for $k \in \mathbf{C}^+$. ■

It is already known[9] that if $Q \in L_1^1(\mathbf{R})$, then the number of bound states for (2.6) is finite; let us denote that number by $\mathscr{N}$, and let $i\kappa_1, \ldots, i\kappa_{\mathscr{N}}$ with $0 < \kappa_1 < \ldots < \kappa_{\mathscr{N}}$ denote the zeros of $1/T^{[0]}(k)$ in $\mathbf{C}^+$. In the following, we generalize the second result of Theorem 9.3 to the case $Q(x) \not\equiv 0$.

Theorem 9.4: Assume $P,Q\in L_1^1(\mathbf{R})$. The k-values in $\mathbf{C}^+$ satisfying $\int_{-\infty}^{\infty}dx\ |ikP(x)+Q(x)|\leqslant 2|k|$ cannot be zeros of $1/T^+(k)$. Moreover, there are no zeros of $1/T^+(k)$ in $\mathbf{C}^+\setminus\{i\kappa_1,\ldots,i\kappa_{\mathcal{N}}\}$ satisfying $|T^{[0]}(k)|\|P\|_{1,1}<2e^{-\|Q\|_{1,1}}$.

Proof: Let $k\in\mathbf{C}^+$ correspond to a bound state of (1.1). We can transform (1.1) into (9.6) with

$$\varphi(k,x)=|ikP(x)+Q(x)|^{1/2}\psi^+(k,x),$$

$$\mathscr{B}(k;x,y)=\frac{1}{2ik}\,e^{ik|x-y|}|ikP(x)+Q(x)|^{1/2}\,\frac{[ikP(y)+Q(y)]}{|ikP(y)+Q(y)|^{1/2}}.$$

A sufficient condition for the absence of bound states is $\|\mathscr{B}\|_{\mathrm{HS}}<1$. Proceeding as in the proof of Theorem 9.3, for $k\in\mathbf{C}^+$ we obtain $\|\mathscr{B}\|_{\mathrm{HS}}<\int_{-\infty}^{\infty}dx\ |ikP(x)+Q(x)|/(2|k|)$, and hence there are no zeros of $1/T^+(k)$ at the k-values in $\mathbf{C}^+$ satisfying $\int_{-\infty}^{\infty}dx\ |ikP(x)+Q(x)|\leqslant 2|k|$. In the special case $P(x)\equiv 0$, this implies that there are no bound states when $|k|>\int_{-\infty}^{\infty}dx\ |Q(x)|/2$. To prove the second part of the theorem, we note that the kernel of the resolvent of the operator $[-d^2/dx^2+Q(x)-k^2]^{-1}$ is given by[20]

$$\mathscr{R}(k;x,y)=\frac{\theta(y-x)f_r^{[0]}(k,x)f_l^{[0]}(k,y)+\theta(x-y)f_l^{[0]}(k,x)f_r^{[0]}(k,y)}{[f_l^{[0]}(k,\cdot);f_r^{[0]}(k,\cdot)]},\tag{9.7}$$

where $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ are the Jost solutions of (2.6) and $\theta(x)$ is the Heaviside function. As seen from (2.8), the Wronskian in (9.7) is equal to $-2ik/T^{[0]}(k)$, and hence we get

$$\|ik|P(x)|^{1/2}\mathscr{R}(k;x,y)P(y)/|P(y)|^{1/2}\|_{\mathrm{HS}}^2=\tfrac{1}{4}|T^{[0]}(k)|^2C(k),\tag{9.8}$$

where we have defined

$$C(k)=\int_{-\infty}^{\infty}dx\ |P(x)||f_l^{[0]}(k,x)|^2\int_{-\infty}^{x}dy\ |f_r^{[0]}(k,y)|^2|P(y)| + \int_{-\infty}^{\infty}dx\ |P(x)||f_r^{[0]}(k,x)|^2\int_{x}^{\infty}dy\ |f_l^{[0]}(k,y)|^2|P(y)|.\tag{9.9}$$

Using (2.1) with $P(x)=0$, we deduce that

$$|f_l^{[0]}(k,x)|\leqslant(1+\max\{0,-x\})e^{-x\ \mathrm{Im}\ k}e^{\int_x^{\infty}dy\ (1+|y|)|Q(y)|},\quad k\in\overline{\mathbf{C}^+},\tag{9.10}$$

$$|f_r^{[0]}(k,x)|\leqslant(1+\max\{0,x\})e^{x\ \mathrm{Im}\ k}e^{\int_{-\infty}^{x}dy\ (1+|y|)|Q(y)|},\quad k\in\overline{\mathbf{C}^+}.\tag{9.11}$$

In (9.9), using (9.10), (9.11), and the estimates $1+\max\{0,\pm x\}\leqslant 1+|x|$ and $e^{\mp 2(x-y)\mathrm{Im}\,k}\leqslant 1$ for $\pm(x-y)\geqslant 0$, we obtain $|C(k)|\leqslant\|P\|_{1,1}^2e^{2\|Q\|_{1,1}}$. Hence the Hilbert–Schmidt norm on the left-hand side of (9.8) is strictly less than 1, provided $|T^{[0]}(k)|<2e^{-\|Q\|_{1,1}}/\|P\|_{1,1}$. Under this condition on k, there is no bound state corresponding to that $k\in\mathbf{C}^+$. ■

Let us denote the number of bound states of (1.1), i.e., the number of zeros of $1/T^+(k)$ in $\mathbf{C}^+$ (including multiplicities) by $N(P,Q)$. In the next two propositions we obtain some stability results for $N(P,Q)$ under certain perturbations of P and Q. As in Sec. VII, we let $T_j^+(k)$ denote the transmission coefficient corresponding to (7.1) for $j=1,2$.

Proposition 9.5: Assume $P_1,P_2\in L^1(\mathbf{R})$, $Q_1,Q_2\in L_1^1(\mathbf{R})$, $1/T_1^+(k)$ does not have any real zeros, and $Q_1(x)$ is a generic potential. If $\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}$ is small, i.e., if (9.13) is satisfied, then

(a) $1/T_2^+(k)$ does not have any real zeros.
(b) $N(P_2,Q_2)=N(P_1,Q_1)$.
(c) If all zeros of $1/T_1^+(k)$ are simple and purely imaginary, so are those of $1/T_2^+(k)$.

Proof: For $a>0$ let Γ_a be the positively oriented contour consisting of the interval $[-a,a]$ and the semicircle $\{k\in\overline{\mathbf{C}^+}:|k|=a\}$, and let us choose a large enough so that all zeros of $1/T_1^+(k)$ in $\mathbf{C}^+$ have an absolute value less than a. Putting $F_j(k)=k/[(k+i)T_j^+(k)]$ for $j=1,2$, from Propositions 7.1 and 7.2 we get

$$\left|\frac{F_1(k)-F_2(k)}{F_1(k)}\right|\leqslant\frac{C}{|F_1(k)|}\,(\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}). \tag{9.12}$$

If $1/T_1^+(k)$ does not have any real zeros, then in the generic case $F_1(k)$ cannot have any real zeros; thus, $F_1(k)$ does not vanish on Γ_a. Moreover, by Proposition 4.1, $F_1(k)$ is continuous and bounded in $\overline{\mathbf{C}^+}$. Hence $\min_{k\in\Gamma_a}|F_1(k)|>0$. Now choosing

$$\|P_1-P_2\|_1+\|Q_1-Q_2\|_{1,1}<\frac{1}{C}\min_{k\in\Gamma_a}\left|\frac{k}{(k+i)T_1^+(k)}\right|, \tag{9.13}$$

from (9.12) we obtain

$$\left|\frac{F_1(k)-F_2(k)}{F_1(k)}\right|<1,\quad k\in\Gamma_a. \tag{9.14}$$

Hence from (9.14) we see $F_2(k)$ cannot vanish on Γ_a, which implies (a). Part (b) follows from (9.14) with the use of Rouché's theorem. Part (c) follows by replacing Γ_a with the union of $N(P_1,Q_1)$ small, positively oriented circles centered at the zeros of $1/T_1^+(k)$ and by applying Rouché's theorem. ■

Proposition 9.6: Assume $Q_1=Q_2=Q$ in (7.1), $1/T_1^+(k)$ does not have any real zeros, $Q(x)$ is an exceptional potential, and $P_1,P_2,Q\in L_1^1(\mathbf{R})$. If $\|P_1-P_2\|_{1,1}$ is small, i.e., if (9.15) is satisfied, then we have

(a) $1/T_2^+(k)$ does not have any real zeros.
(b) $N(P_2,Q)=N(P_1,Q)$.
(c) If all zeros of $1/T_1^+(k)$ are simple and purely imaginary, so are those of $1/T_2^+(k)$.

Proof: We will proceed as in the proof of Proposition 9.5. Let us choose Γ_a as in that proof but define $F_j(k)=1/T_j^+(k)$ for $j=1,2$, instead. Note that $F_1(k)$ is bounded, continuous, and nonzero on Γ_a. From (7.13) we have

$$\left|\frac{F_1(k)-F_2(k)}{F_1(k)}\right|\leqslant C\|P_1-P_2\|_{1,1}.$$

From (9.8) we get

$$\left|\frac{F_1(k)-F_2(k)}{F_1(k)}\right|\leqslant\frac{C}{|F_1(k)|}\,\|P_1-P_2\|_{1,1},$$

and hence by choosing

$$\|P_1-P_2\|_{1,1}<\frac{1}{C}\min_{k\in\Gamma_a}\left|\frac{1}{T_1^+(k)}\right|, \tag{9.15}$$

and proceeding as in the proof of Proposition 9.5, we complete the proof. ■

Theorem 9.7: Assume $P,Q\in L^1(\mathbf{R})$ and $P(x)\leqslant 0$. Then, either $N(P,Q)$ and $N(0,Q)$ are both infinite, or they are both finite and $N(P,Q)=N(0,Q)$. Thus the number of bound states of (1.1) coincides with the number of bound states of (2.6).

Proof: Since $P(x)\leqslant 0$, by Corollary 9.2 we know that the bound states of (1.1) can only occur when k is on the positive imaginary axis. Let us write (1.1) with $k=i\beta$ as two simultaneous equations:

$$-\psi''+V(\beta,x)\psi=E(\beta)\psi, \tag{9.16}$$

$$E(\beta)=-\beta^2, \tag{9.17}$$

where β is considered to be a parameter in the potential $V(\beta,x)=Q(x)-\beta P(x)$ of the Schrödinger equation (9.16), and $E(\beta)$ denotes the corresponding energy for each β. Each bound-state energy $-\kappa_j^2$ of (2.6) gives rise to an eigenvalue branch $E_j(\beta)$. From (9.16) we have

$$E'(\beta)=\frac{\langle\psi,-P\psi\rangle}{\langle\psi,\psi\rangle}, \tag{9.18}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $L^2(\mathbf{R})$. If $P(x)\leqslant 0$, from (9.18) we see that $E'(\beta)\geqslant 0$ and hence each $E_j(\beta)$ is a nondecreasing function of β. Therefore for $\beta\geqslant 0$, the graph of $E_j(\beta)$ must intersect the parabola $E=-\beta^2$ at exactly one point, say $(\beta_j,-\beta_j^2)$, and each $E_j(\beta)$ gives rise to exactly one solution of (9.17). The number $N(P,Q)$ is equal to the number of intersections of the eigenvalue branches $E_j(\beta)$ with the parabola given in (9.17) for $j\geqslant 1$. Since each of the $N(0,Q)$ branches is responsible for exactly one intersection, we conclude that $N(P,Q)=N(0,Q)$. Note that if $Q\in L^1(\mathbf{R})$ but $Q\notin L_1^1(\mathbf{R})$, it is possible that $N(0,Q)=+\infty$, but then we also have $N(P,Q)=+\infty$. ■

If $P,Q\in L^1(\mathbf{R})$ and $P(x)\leqslant 0$, then either $N(P,Q)=+\infty$, in which case the set of bound-state energies of (1.1) consists of a strictly decreasing sequence of negative numbers converging to 0, or $N(P,Q)$ is finite and equal to $\mathcal{N}$, in which case we let $k=i\beta_j$ for $j=1,...,\mathcal{N}$, with $0<\beta_1<\ldots<\beta_{\mathcal{N}}$ denote the zeros of $1/T^+(k)$ in $\mathbf{C}^+$. Since the condition $Q\in L_1^1(\mathbf{R})$ guarantees the finiteness of $N(0,Q)$, from Theorem 9.1 we obtain the following:

Corollary 9.8: Assume $P(x)\leqslant 0$, $Q(x)\geqslant 0$, $P\in L^1(\mathbf{R})$, and $Q\in L_1^1(\mathbf{R})$. Then, there are no zeros of $1/T^+(k)$ in $\overline{\mathbf{C}^+}$.

In the next theorem, when $P(x)\leqslant 0$, using the constant $P_{\min}$ defined in (9.1), we obtain some upper and lower bounds on each bound-state energy of (1.1).

Theorem 9.9: Assume $N(0,Q)$ is finite and nonzero, $P_{\min}$ is finite, $P(x)\leqslant 0$, and $P,Q\in L^1(\mathbf{R})$; let $k=i\kappa_j$ correspond to the bound states of (2.6) for $j=1,...,\mathcal{N}$. Then, the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ occur at $k=i\beta_j$ satisfying $\beta_*\leqslant\beta_j\leqslant\kappa_j$ for $j=1,...,\mathcal{N}$, where $\beta_*=P_{\min}/2+\sqrt{P_{\min}^2/4+\kappa_1^2}$. In particular, $\beta_1\geqslant\beta_*$ and $\beta_{\mathcal{N}}\leqslant\kappa_{\mathcal{N}}$, with the equalities holding if and only if $P(x)=0$.

Proof: At a bound state with $k=i\beta_j$ of (1.1), replacing k_0 in (9.5) by $0+i\beta_j$, we get

$$\int_{-\infty}^{\infty}dx\; f_l^{+\prime}(i\beta_j,x)^2=\int_{-\infty}^{\infty}dx\;[-\beta_j^2+\beta_jP(x)-Q(x)]f_l^+(i\beta_j,x)^2. \tag{9.19}$$

On the other hand, since $-\kappa_{\mathcal{N}}^2$ is the lowest bound-state energy for (2.6), we have

$$-\kappa_{\mathcal{N}}^2\leqslant\frac{\int_{-\infty}^{\infty}dx\;[f_l^{+\prime}(i\beta_j,x)^2+Q(x)f_l^+(i\beta_j,x)^2]}{\int_{-\infty}^{\infty}dx\; f_l^+(i\beta_j,x)^2}, \tag{9.20}$$

with the equality holding if and only if $f_l^{[0]}(i\kappa_{\mathcal{N}},x)$ and $f_l^+(i\beta_j,x)$ are linearly dependent. From (9.19) and (9.20) we obtain

$$-\kappa_{\mathcal{N}}^2\leqslant-\beta_j^2+\beta_j\frac{\int_{-\infty}^{\infty}dx\; P(x)f_l^+(i\beta_j,x)^2}{\int_{-\infty}^{\infty}dx\; f_l^+(i\beta_j,x)^2}. \tag{9.21}$$

Since $P(x)\leqslant 0$, from (9.21) we see that $\beta_j\leqslant\kappa_{\mathcal{N}}$ with the equality holding if and only if $P(x)=0$ and $j=\mathcal{N}$. Thus $\beta_j\in(0,\kappa_{\mathcal{N}}]$ for $j=1,...,\mathcal{N}$. Now let us improve the bounds on β_j. From the proof of Theorem 9.7, recall that each eigenvalue branch $E_j(\beta)$ gives rise to exactly one solution of (9.17) starting with $-\kappa_j^2$ at $\beta=0$ and ending with $-\beta_j^2$ at $\beta=\beta_j$. Since $E_j(\beta)$ is an increasing function of β, we get $-\kappa_j^2=E_j(0)\leqslant E_j(\beta_j)=-\beta_j^2$, and hence $\beta_j\leqslant\kappa_j$. Now consider $E_1(\beta)$, the eigenvalue branch corresponding to β_1. From (9.18) using $P_{\min}\leqslant P(x)$, we obtain $0\leqslant E_1'(\beta)\leqslant-P_{\min}$; more specifically, $0<E_1'(\beta)<-P_{\min}$ unless $P(x)=0$. Since $E_1(0)=-\kappa_1^2$ and $E_1(\beta)$ is nondecreasing, we get $E_1(\beta)\leqslant-\beta P_{\min}-\kappa_1^2$. Thus from the inequality $-\beta_1^2\leqslant-\beta_1P_{\min}-\kappa_1^2$, we get $\beta_1\geqslant\beta_*$. Note that the equality in $\beta_1\geqslant\beta_*$ holds if and only if $P(x)=0$ because $\beta_*\geqslant\kappa_1$ with the equality holding if and only if $P(x)\geqslant 0$. ■

From the proof of Theorem 9.9 we get the following corollary that holds even when (2.6) has infinitely many bound states.

Corollary 9.10: Assume $P,Q\in L^1(\mathbf{R})$ and $P(x)\leqslant 0$, and let $\{\mathscr{E}_j\}$ and $\{\mathscr{E}_j^{[0]}\}$ for $j\geqslant 1$ denote the bound-state energies of (1.1) and (2.6), respectively, ordered such that $\mathscr{E}_j<\mathscr{E}_{j+1}$ and $\mathscr{E}_j^{[0]}<\mathscr{E}_{j+1}^{[0]}$. Then we have $\mathscr{E}_j^{[0]}\leqslant\mathscr{E}_j<0$ for $j\geqslant 1$, and hence the bound-state energies of (1.1) cannot occur below the lowest bound-state energy of (2.6).

Recall that the Levinson theorem[21] relates the number of bound states for the usual Schrödinger equation to the change in the phase of the transmission coefficient. Next we generalize the Levinson theorem to (1.1).

Theorem 9.11: Assume that $P\in L^1(\mathbf{R})$ in the generic case and $P\in L_1^1(\mathbf{R})$ in the exceptional case and that $Q\in L_1^1(\mathbf{R})$, and suppose $1/T^+(k)$ does not have any real zeros. Then the number of bound states of (1.1) is given by

$$\arg\, T^+(0+)=\pi\left[N(P,Q)-\frac{d}{2}\right], \tag{9.22}$$

where $d=0$ in the exceptional case and $d=1$ in the generic case, and $\arg T^+(k)$ denotes the continuous branch of the argument of $T^+(k)$ normalized such that $\arg T^+(+\infty)=0$.

Proof: For $b>a>0$, let $\Gamma_{a,b}$ be the positively oriented contour consisting of the circular arcs $\{k\in\overline{\mathbf{C}^+}:|k|=a\}$ and $\{k\in\overline{\mathbf{C}^+}:|k|=b\}$ and the segments $[-b,-a]$ and $[a,b]$. Let us choose a and b so that all zeros of $1/T^+(k)$ in $\mathbf{C}^+$ are enclosed by $\Gamma_{a,b}$. By the argument principle we have

$$N(P,Q)=-\frac{1}{2\pi i}\int_{\Gamma_{a,b}}dk\,\frac{\dot{T}^+(k)}{T^+(k)}=-\frac{1}{2\pi}\,\Delta_{\Gamma_{a,b}}[\arg\, T^+(k)], \tag{9.23}$$

where $\Delta_{\Gamma_{a,b}}[\arg T^+(k)]$ indicates the change in the argument of $T^+(k)$ when $\Gamma_{a,b}$ is traversed once. This change is independent of a and b, and hence we evaluate it by letting $a\to 0$ and $b\to+\infty$. By Theorem 6.1 the contribution to that change from the large semicircle $\{k\in\overline{\mathbf{C}^+}:|k|=b\}$ vanishes as $b\to+\infty$. In view of (5.2) and (5.3), we see that the contribution from the small semicircle $\{k\in\overline{\mathbf{C}^+}:|k|=a\}$ in the limit $a\to 0$ is equal to 0 in the exceptional case and $-\pi$ in the generic case. Thus the contribution from the interval $(0,+\infty)$ is given by $\arg T^+(+\infty)-\arg T^+(0+)=-\arg T^+(0+)$. By the first equality in (4.10), the contribution from the interval $(-\infty,0)$ is the same. Hence, the right-hand side in (9.23) is equal to $(1/\pi)\arg T^+(0+)$ in the exceptional case and $(1/2)+(1/\pi)\arg T^+(0+)$ in the generic case, which gives us (9.22). ■

Finally, let us show that in the special case when P and Q have support in a half-line, we can relate the poles of the transmission coefficient to the poles of a reflection coefficient. Since there is no loss of generality in choosing our half-lines as $\mathbf{R}^+=(0,+\infty)$ or $\mathbf{R}^-=(-\infty,0)$ instead of $(a,+\infty)$ or $(-\infty,b)$, respectively, for some constants a and b, we will state the following proposition using $\mathbf{R}^\pm$.

Proposition 9.12: Assume $P(x)=Q(x)=0$ for $x\in\mathbf{R}^-$ and $P,Q\in L^1(\mathbf{R}^+)$. Then $L^+(k)$ is meromorphic in $\mathbf{C}^+$ having poles coinciding with the poles of $T^+(k)$. Furthermore, none of the zeros of $L^+(k)$ coincide with the poles of $T^+(k)$ in $\mathbf{C}^+$. These assertions remain valid if $\mathbf{R}^-$ and $L^+(k)$ are replaced by $\mathbf{R}^+$ and $R^+(k)$, respectively.

Proof: If $P(x)=Q(x)=0$ for $x\in\mathbf{R}^-$, from Theorem 2.1 (i), we see that $f_l^+(k,0)$ and $f_l^{+\prime}(k,0)$ are analytic in $\mathbf{C}^+$. Hence using (1.3) we can conclude that $L^+(k)/T^+(k)$ is analytic in $\mathbf{C}^+$, allowing us to conclude that the poles of $L^+(k)$ and $T^+(k)$ must coincide in $\mathbf{C}^+$. Since $f_l^+(k,0)$ and $f_l^{+\prime}(k,0)$ cannot vanish simultaneously, it follows that $1/T^+(k)$ and $L^+(k)/T^+(k)$ cannot vanish simultaneously in $\mathbf{C}^+$, and hence the zeros of $L^+(k)$ and the poles of $T^+(k)$ cannot coincide in $\mathbf{C}^+$. The proof when P and Q have support in $\mathbf{R}^-$ is obtained in a similar manner. ■

X. EIGENVALUE CURVES AND ZEROS OF JOST SOLUTIONS

In this section we study the zeros of the Jost solutions of (1.1) for a fixed $k\in\overline{\mathbf{C}^+}$ and analyze the number of such zeros in relation to the bound states of (1.1) and (2.6). As in Sec. IX, we let $N(P,Q)$ denote the number of bound states of (1.1). When $P(x)\leqslant 0$ we show that the number of zeros of the Jost solutions of (1.1) is related to $N(P,Q)$ in a simple manner, and we present some examples showing that this relation does not hold in general. We establish the connection between the results of Sec. VIII on Jordan chains and certain zeros of the Jost solutions of (1.1). This connection uses the eigenvalue branches introduced in the proof of Theorem 9.7. We also show that the number of bound states of (1.1) with real energies is greater than or equal to $N(0,Q)$.

In the first proposition, we collect some results about the oscillation properties of solutions of generalized Schrödinger equations related by inequalities involving the coefficients. Although the methods for proving such results are familiar,[20,22] we include a proof for the convenience of the reader.

Consider the pair of generalized Schrödinger equations

$$\chi_j''(\mu,x)-\mu^2\chi_j(\mu,x)=V_j(\mu,x)\chi_j(\mu,x),\quad \mu\geqslant 0,\quad j=1,2. \tag{10.1}$$

Note that if we let $V_j(\mu,x)=-\mu P(x)+Q(x)$ in (10.1), we get (1.1) for $k=i\mu$.

Proposition 10.1: Assume $V_j(\mu,\cdot)\in L^1(\mathbf{R})$ if $\mu>0$, $V_j(0,\cdot)\in L_1^1(\mathbf{R})$, and $V_1(\mu_2,x)\leqslant V_2(\mu_1,x)$ if $0\leqslant\mu_1\leqslant\mu_2$. Let $\chi_1(\mu_1,x)$ and $\chi_2(\mu_2,x)$ denote two nontrivial solutions of (10.1) with the corresponding coefficients $V_1(\mu_1,x)$ and $V_2(\mu_2,x)$, respectively. Then:

(i) Suppose $\chi_2(\mu_2,x)$ has two successive zeros a and b with $a<b$. If $0\leqslant\mu_1<\mu_2$, then $\chi_1(\mu_1,x)$ has at least one zero in (a,b). If $0\leqslant\mu_1=\mu_2$ and $V_1(\mu_1,x)\not\equiv V_2(\mu_1,x)$ on (a,b), then $\chi_1(\mu_1,x)$ has at least one zero in (a,b). If $0\leqslant\mu_1=\mu_2$, $V_1(\mu_1,x)\equiv V_2(\mu_1,x)$ on (a,b), and $\chi_1(\mu_1,x)$ and $\chi_2(\mu_1,x)$ are linearly independent in (a,b), then $\chi_1(\mu_1,x)$ has exactly one zero in (a,b).

(ii) Suppose $\chi_2(\mu_2,x)$ remains bounded as $x\to+\infty$. Let a denote the largest zero of $\chi_2(\mu_2,x)$, and set $b=+\infty$. Then the assertions of (i) remain true if we replace the interval (a,b) by $(a,+\infty)$.

(iii) If $0\leqslant\mu_1\leqslant\mu_2$, $\chi_2(\mu_2,x)$ is bounded as $x\to+\infty$, and $\chi_1(\mu_1,x)$ has no zeros in $\mathbf{R}$, then $\chi_2(\mu_2,x)$ has no zeros in $\mathbf{R}$ either.

(iv) If $\chi_2(\mu_2,x)$ is bounded as $x\to-\infty$ and a is the smallest zero of $\chi_2(\mu_2,x)$, then the assertion of (iii) holds, and the assertions in (i) remain true if we replace the interval (a,b) by $(-\infty,a)$.

Proof: We will omit the proof of (i) because on a finite interval such results are known (e.g., Theorem 1.1 on p. 208 of Ref. 20). Moreover, our proof of (ii) is easily modified to prove (i). The proof of (iv) is analogous to the proofs of (ii) and (iii), and hence we will only prove (ii) and (iii).

(ii) The proof can be given using contradiction. Without loss of generality we may assume that $\chi_1(\mu_1,x)$ and $\chi_2(\mu_2,x)$ are strictly positive in $(a,+\infty)$. When $b>a$, where a is the largest zero of $\chi_2(\mu_2,x)$, from (10.1) we get

$$\chi_2(\mu_2,b)\chi_1'(\mu_1,b)-\chi_2'(\mu_2,b)\chi_1(\mu_1,b)+\chi_2'(\mu_2,a)\chi_1(\mu_1,a)$$
$$=\int_a^b dx\,[V_1(\mu_1,x)-V_2(\mu_2,x)+\mu_1^2-\mu_2^2]\chi_1(\mu_1,x)\chi_2(\mu_2,x). \tag{10.2}$$

Note that, by the asymptotic properties of the solutions and their assumed positivity in the interval $(a,+\infty)$, we have for $\mu_2>0$ and some $c_2>0$

$$\chi_2(\mu_2,x)=c_2e^{-\mu_2x}+o(e^{-\mu_2x})\quad \chi_2'(\mu_2,x)=-c_2\mu_2e^{-\mu_2x}+o(e^{-\mu_2x}),\quad x\to+\infty. \tag{10.3}$$

Furthermore, if $\chi_1(\mu_1,x)$ is unbounded as $x\to+\infty$, then for some $c_1>0$ we have

$$\chi_1(\mu_1,x)=c_1e^{\mu_1x}+o(e^{\mu_1x}),\quad \chi_1'(\mu_1,x)=c_1\mu_1e^{\mu_1x}+o(e^{\mu_1x}),\quad x\to+\infty. \tag{10.4}$$

If $\mu_1=0$ and $\chi_1(0,x)$ is unbounded as $x\to+\infty$, then for some $\tilde{c}_1>0$

$$\chi_1(0,x)=\tilde{c}_1x+o(x),\quad \chi_1'(0,x)=\tilde{c}_1+o(1),\quad x\to+\infty. \tag{10.5}$$

If $\chi_2(0,x)$ is bounded as $x\to+\infty$, then for some $\tilde{c}_2>0$ we have

$$\chi_2(0,x)=\tilde{c}_2+o(1), \quad \chi_2'(0,x)=o(1/x), \quad x\to+\infty. \tag{10.6}$$

Using (10.3)–(10.6) we will let $b\to+\infty$ in (10.2). When $0\leqslant\mu_1\leqslant\mu_2$, the limit as $b\to+\infty$ of the right-hand side of (10.2) exists and is nonpositive; it is equal to zero precisely when $\mu_1=\mu_2$ and $V_1(\mu_1,x)\equiv V(\mu_2,x)$ on $(a,+\infty)$. If $0\leqslant\mu_1<\mu_2$, the limit of the left-hand side of (10.2) is equal to $\chi_2'(\mu_2,a)\chi_1(\mu_1,a)$, which is nonnegative. Hence we have a contradiction, and thus $\chi_1(\mu_1,x)$ must have a zero in $(a,+\infty)$. If $0<\mu_1=\mu_2$ and $\chi_1(\mu_1,x)$ is unbounded, then the limit on the left-hand side of (10.2) is equal to $2c_1c_2\mu_1+\chi_2'(\mu_1,a)\chi_1(\mu_1,a)$, which is strictly positive; the right-hand side is nonpositive and so again we have a contradiction. If $0<\mu_1=\mu_2$ and $\chi_1(\mu_1,x)$ is bounded, then the limit of the left-hand side is equal to $\chi_2'(\mu_2,a)\chi_1(\mu_1,a)$, which is nonnegative. If also $V_1(\mu_1,x)\not\equiv V_2(\mu_2,x)$ on $(a,+\infty)$, then the right-hand side is strictly negative and we have a contradiction. If $V_1(\mu_1,x)\equiv V_2(\mu_2,x)$ on $(a,+\infty)$, then $\chi_1(\mu_1,a)>0$ due to the linear independence of $\chi_1(\mu_1,x)$ and $\chi_2(\mu_2,x)$, and so the left-hand side of (10.2) is strictly positive while its right-hand side is zero. In this case, by (i), there can only be one zero of $\chi_1(\mu_1,x)$ in $(a,+\infty)$. If $0=\mu_1=\mu_2$ and $\chi_1(0,x)$ is unbounded, then because of (10.6) the left-hand side of (10.2) approaches $\tilde{c}_1\tilde{c}_2+\chi_2'(0,a)\chi_1(0,a)$, which is again strictly positive. If $\mu_2=\mu_1=0$ and $\chi_1(0,x)$ is bounded, then the limit of the left-hand side of (10.2) is $\chi_2'(0,a)\chi_1(0,a)$, which is nonnegative. If $V_1(0,x)\not\equiv V_2(0,x)$ on $(a,+\infty)$, then the right-hand side of (10.2) is strictly negative, while if $V_1(0,x)\equiv V_2(0,x)$ on $(a,+\infty)$, then its right-hand side is zero and its left-hand side is strictly positive due to the linear independence of $\chi_1(0,x)$ and $\chi_2(0,x)$. In both cases we arrive at a contradiction. As in (i), if $V_1(0,x)\equiv V_2(0,x)$ on $(a,+\infty)$, we conclude that there is exactly one zero of $\chi_1(0,x)$ in $(a,+\infty)$.

(iii) Suppose $\chi_2(\mu_2,x)$ does have some zeros, the largest of which is a. Then, under the assumptions made in (i) and (ii), it follows that $\chi_1(\mu_1,x)$ has a zero to the right of a, contradicting the assumptions of (iii). The only situation not covered by (i) and (ii) is when $\mu_2=\mu_1$, $V_1(\mu_1,x)\equiv V_2(\mu_2,x)$ on $(a,+\infty)$, and $\chi_1(\mu_1,x)$ and $\chi_2(\mu_1,x)$ are linearly dependent in $(a,+\infty)$, but then $\chi_2(\mu_1,a)=0$ implies $\chi_1(\mu_1,a)=0$, which is again a contradiction. ■

From Proposition 10.1 we obtain the following:

Corollary 10.2: Assume $P,Q\in L^1(\mathbf{R})$ and let $\beta>0$. Suppose that $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ are linearly independent. Then $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros and their zeros are separated, i.e., between two successive zeros of $f_l^+(i\beta,x)$ there is a zero of $f_r^+(i\beta,x)$ and vice versa. Moreover, to the right of the largest zero of $f_l^+(i\beta,x)$ there is a zero of $f_r^+(i\beta,x)$, and to the left of the smallest zero of $f_r^+(i\beta,x)$ there is a zero of $f_l^+(i\beta,x)$.

Our next result concerns the zeros of the Jost solutions of (2.6). Since some theorems of this type have already been proved elsewhere (see, e.g., Theorem 14.10 of Ref. 22 or Theorem XIII.8 on p. 90 of Ref. 23), we only comment on certain details that may not be obvious from those references. Recall that $N(0,Q)$ denotes the number of bound states of (2.6).

Proposition 10.3: (i) Suppose $Q\in L^1(\mathbf{R})$ and assume $\beta>0$. Then the number of zeros of $f_l^{[0]}(i\beta,x)$ is equal to the number of bound states of (2.6) with energies contained in the interval $(-\infty,-\beta^2)$.

(ii) Suppose further that $Q\in L_1^1(\mathbf{R})$. Then the number of zeros of $f_l^{[0]}(0,x)$ is equal to $N(0,Q)$.

Proof: (i) Since we only assume $Q\in L^1(\mathbf{R})$, there may be infinitely many bound states of (2.6) with energies accumulating at zero. All such energies are negative, and let us denote them by $-\gamma_j^2$ with $\gamma_j>\gamma_{j+1}>0$ for $j\geqslant1$. It is known (Theorem 14.10 of Ref. 22) that $f_l^{[0]}(i\gamma_j,x)$ has exactly $(j-1)$ zeros. Hence we only need to consider the zeros of $f_l^{[0]}(i\beta,x)$ when β is not equal to any γ_j. If $\beta>\gamma_1$, then from Proposition 10.1 (iii) with $V_1=V_2=Q$, $\mu_1=\gamma_1$, $\mu_2=\beta$, $\chi_1(\mu_1,x)=f_l^{[0]}(i\gamma_1,x)$, and $\chi_2(\mu_2,x)=f_l^{[0]}(i\beta,x)$, it follows that $f_l^{[0]}(i\beta,x)$ has no zeros. If $\beta\in(\gamma_{j+1},\gamma_j)$, then, by Proposition 10.1 (i) and (ii) with $\mu_1=\beta$ and $\mu_2=\gamma_j$, we observe that $f_l^{[0]}(i\beta,x)$ has at least j zeros. On the other hand, using Proposition 10.1, we can conclude that $f_l^{[0]}(i\beta,x)$ cannot have more than j zeros because the number of its zeros is nondecreasing as β decreases and $f_l^{[0]}(i\gamma_{j+1},x)$ has exactly j zeros. Thus $f_l^{[0]}(i\beta,x)$ has exactly j zeros when $\beta\in(\gamma_{j+1},\gamma_j)$. This proves (i) when $N(0,Q)=+\infty$ because the bound states of (2.6) can only occur when k is on the positive imaginary axis. If $N(0,Q)$ is finite and is denoted by $\mathcal{N}$, then we must still consider the case when $\beta\in(0,\gamma_{\mathcal{N}})$. Then using Lemma 1 on p. 91 of Ref. 23 we conclude that $f_l^{[0]}(i\beta,x)$ has exactly $\mathcal{N}$ zeros because if it had more than $\mathcal{N}$ zeros one could find a subspace of dimension at least $\mathcal{N}+1$ on which the expectation value of $(-d^2/dx^2+Q-\beta P)$ is less than or equal to $-\beta^2$, and this would imply the existence of at least $(\mathcal{N}+1)$ eigenvalues less than or equal to $-\beta^2$.

(ii) In this case the condition $Q\in L_1^1(\mathbf{R})$ guarantees that $N(0,Q)$ is finite. It only remains to consider the case $\beta=0$. Note that $f_l^{[0]}(0,x)$ cannot have more than $\mathcal{N}$ zeros; this is because $f_l^{[0]}(i\beta,x)$ has exactly $\mathcal{N}$ zeros when β is sufficiently small and by (2.5) we see that as $\beta\to 0$ we have $f_l^{[0]}(i\beta,x)\to f_l^{[0]}(0,x)$ uniformly on compact x-intervals. On the other hand, in (10.1) by setting $\mu_1=0$, $\mu_2=\beta$, $V_1(\mu_1,x)=Q(x)$, $V_2(\mu_2,x)=Q(x)-\beta P(x)$, $V_1(\mu_1,x)=Q(x)$, and $\chi_2(\mu_2,x)=f_l^{[0]}(\beta,x)$, and using Proposition 10.1, we see that $f_l^{[0]}(0,x)$ has at least $\mathcal{N}$ zeros. Hence $f_l^{[0]}(0,x)$ must have exactly $\mathcal{N}$ zeros. ■

If $Q\in L_1^1(\mathbf{R})$, then $N(0,Q)$ is finite and as in Sec. IX we let $k=i\kappa_j$ for $j=1,...,\mathcal{N}$ denote the bound states of (2.6). From Theorems 9.7 and 9.9, when $P\in L^1(\mathbf{R})$, $Q\in L_1^1(\mathbf{R})$, and $P(x)\leqslant 0$, we already know that the bound states of (1.1) occur at $k=i\beta_j$ satisfying $\beta_j\leqslant\kappa_j$ for $j=1,...,\mathcal{N}$. In the next theorem, we extend Proposition 10.3 to (1.1) and analyze the number of zeros of the Jost solutions of (1.1) when k is on the positive imaginary axis.

Theorem 10.4: Assume that $P\in L^1(\mathbf{R})$, $Q\in L_1^1(\mathbf{R})$, and $P(x)\leqslant 0$. Then, for each $\beta\geqslant 0$, the functions $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros, and this number is equal to the number of bound states of (1.1) with energies contained in the interval $(-\infty,-\beta^2)$.

Proof: From Proposition 10.1 (i) and (ii), we see that $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros. Since $P(x)\leqslant 0$, from the proof of Theorem 9.7 it follows that, for any fixed $\beta>0$, the number of eigenvalues of the operator $(-d^2/dx^2+Q-\beta P)$ below $-\beta^2$ is equal to the number of $E_j(\beta)$-values that lie below $-\beta^2$. Note that if $\beta\in[\beta_j,\beta_{j+1})$ for $j=1,...,\mathcal{N}-1$, then the $(\mathcal{N}-j)$ values $E_{\mathcal{N}}(\beta),E_{\mathcal{N}-1}(\beta),\ldots,E_{j+1}(\beta)$ lie strictly below $-\beta^2$; if $\beta\in[\beta_{\mathcal{N}},+\infty)$ then there are no eigenvalues below $-\beta^2$, and if $\beta\in[0,\beta_1)$ then exactly $\mathcal{N}$ eigenvalues lie below $-\beta^2$. Using Proposition 10.3 when the potential $Q(x)$ in (2.6) is replaced by $Q(x)-\beta P(x)$, we conclude that $f_l^+(i\beta,x)$ has no zeros for $\beta\in[\beta_{\mathcal{N}},+\infty)$, $\mathcal{N}$ zeros for $\beta\in[0,\beta_1)$, and $(\mathcal{N}-j)$ zeros for $\beta\in[\beta_j,\beta_{j+1})$ for $j=1,...,\mathcal{N}-1$. ■

If we weaken the condition $P(x)\leqslant 0$ in Theorem 10.4, the number of bound states may not be easily related to the number of zeros of the Jost solutions, as we will see in Example 11.2. From Proposition 10.1 and Theorem 10.4 we have the following:

Corollary 10.5: Assume that $P,Q\in L^1(\mathbf{R})$ and that $P(x)\leqslant 0$. The zeros of $f_l^+(i\beta_j,x)$ separate the zeros of $f_l^+(i\beta_{j+1},x)$, i.e., between two consecutive zeros of $f_l^+(i\beta_j,x)$ there is exactly one zero of $f_l^+(i\beta_{j+1},x)$, where $k=i\beta_j$ for $j=1,...,\mathcal{N}$ correspond to the bound states of (1.1). Similarly, the zeros of $f_r^+(i\beta_j,x)$ separate the zeros of $f_r^+(i\beta_{j+1},x)$.

When we no longer have $P(x)\leqslant 0$, then there may be bound states of (1.1) with complex energies and $N(P,Q)$ may be larger than $N(0,Q)$. We refer the reader to Examples 11.2 and 11.4. In the next theorem, when $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$, we analyze the bound states of (1.1) when k is on the positive imaginary axis, establish the connection between Theorem 8.4 and the zeros of $f_r^+(i\beta,\,x)$, and also consider multiple zeros of $1/T^+(k)$ on the positive imaginary axis.

Theorem 10.6: Suppose $P\in L^1(\mathbf{R})$ and $Q\in L_1^1(\mathbf{R})$. Then:

(i) If (2.6) has $\mathcal{N}$ bound states with $\mathcal{N}\geqslant 1$, then (1.1) has at least $\mathcal{N}$ bound states with real (negative) energies.

(ii) $1/T^+(i\beta)$ has a zero of order m at some positive β_0 if and only if the function $E_0(\beta)+\beta^2$ has a zero of order m at β_0, where $E_0(\beta)$ denotes the unique eigenvalue branch of the operator $(-d^2/dx^2+Q-\beta P)$ satisfying $E_0(\beta)\to-\beta_0^2$ as $\beta\to\beta_0$. If $m=1$, then the graph of $E_0(\beta)$ and the graph of the parabola $E=-\beta^2$ intersect with different slopes at β_0. If $m\geqslant 2$ and m is even, then the graphs touch at β_0 but do not cross each other. If $m\geqslant 3$ and m is odd, then the graphs cross smoothly such that at the point of intersection they have the same slope.

(iii) The lowest eigenvalue branch $E_{\mathcal{N}}(\beta)$ satisfies $E''_{\mathcal{N}}(\beta)<0$ for $\beta>0$ unless $P(x)\equiv 0$, in which case $E_{\mathcal{N}}(\beta)$ is a constant. Hence the graph of the lowest eigenvalue branch is concave down if $P(x)\not\equiv 0$.

(iv) The number of zeros of $f_l^+(i\beta,x)$ behaves in the following manner as β is increased from $\beta_0-\epsilon$ to $\beta_0+\epsilon$ when ϵ is sufficiently small: If m is even, then the number of zeros is either constant throughout the interval $(\beta_0-\epsilon,\beta_0+\epsilon)$ or it is constant in $(\beta_0-\epsilon,\beta_0)\cup(\beta_0,\beta_0+\epsilon)$ but one less at β_0. If m is odd, then the number of zeros either increases or decreases by one as β crosses β_0. The number of zeros of $f_l^+(i\beta,x)$ can only change at β-values corresponding to bound states of (1.1).

Proof: (i) Since we are only interested in the bound states corresponding to $k=i\beta$ for $\beta>0$, we first derive a lower bound for $E_{\mathcal{N}}(\beta)$, which will show that $E_{\mathcal{N}}(\beta)=\mathrm{o}(\beta^2)$ as $\beta\to+\infty$. Let us indicate the Fourier transform by a caret:

$$\hat{\psi}(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\; e^{iqx}\psi(x),\quad \psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\; e^{-iqx}\hat{\psi}(q).$$

Then, letting $\|\cdot\|_2$ denote the norm on $L^2(\mathbf{R})$, for $a>0$ we get (cf. Theorem IX.28 of Ref. 24)

$$|\psi(x)|\leqslant\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\;|\hat{\psi}(q)|\leqslant\frac{1}{\sqrt{2a}}\left[\int_{-\infty}^{\infty}dq\;(q^2+a^2)|\hat{\psi}(q)|^2\right]^{1/2}=\frac{1}{\sqrt{2a}}\,(\|\psi'\|_2^2+a^2\|\psi\|_2^2)^{1/2},\tag{10.7}$$

where we have used the Schwarz inequality, $\|\psi\|_2=\|\hat{\psi}\|_2$, $\int_{-\infty}^{\infty}dq/(a^2+q^2)=\pi/a$, and $\hat{\psi}'=-iq\hat{\psi}$. Next we use (10.7) to estimate the quadratic forms $\langle Q\psi,\psi\rangle$ and $\langle P\psi,\psi\rangle$. From (10.7) we obtain

$$\int_{-\infty}^{\infty}dx\;|Q(x)||\psi(x)|^2\leqslant\frac{1}{2a}\,(\|\psi'\|_2^2+a^2\|\psi\|_2^2)\left(\int_{-\infty}^{\infty}dx|Q(x)|\right).\tag{10.8}$$

If $P(x)\equiv0$, then (1.1) and (2.6) become identical, and in this trivial case both equations have exactly $\mathcal{N}$ bound states. Thus there is no loss of generality in assuming $P(x)\not\equiv0$. In order to estimate the integral $\int_{-\infty}^{\infty}dx\;|P(x)||\psi(x)|^2$, we split it into two parts: one over the region $\{x:|P(x)|>M\}$ and the other over the region $\{x:|P(x)|\leqslant M\}$, where the constant $M\geqslant0$ is arbitrary for the moment but will be fixed later. Then, for any $b>0$, (10.7) implies

$$\int_{-\infty}^{\infty}dx\;|P(x)||\psi(x)|^2\leqslant\frac{1}{2b}\,(\|\psi'\|_2^2+b^2\|\psi\|_2^2)\left(\int_{\{|P(x)|>M\}}dx\;|P(x)|\right)+M\|\psi\|_2^2.\tag{10.9}$$

Combining (10.8) and (10.9) we get

$$\langle-\psi''+Q\psi-\beta P\psi,\psi\rangle\geqslant J_1\|\psi'\|_2^2-J_2\|\psi\|_2^2,$$

where

$$J_1=1-\frac{1}{2a}\int_{-\infty}^{\infty}dx\;|Q(x)|-\frac{\beta}{2b}\int_{\{|P(x)|>M\}}dx\;|P(x)|,$$

$$J_2=\frac{a}{2}\int_{-\infty}^{\infty}dx\;|Q(x)|+\frac{\beta b}{2}\int_{\{|P(x)|>M\}}dx\;|P(x)|+\beta M.$$

We now set

$$a=\int_{-\infty}^{\infty}dx\;|Q(x)|,\quad b=\beta\int_{\{|P(x)|>M\}}dx\;|P(x)|,$$

and assume that ψ is a normalized eigenfunction corresponding to the eigenvalue $E_{\mathcal{N}}(\beta)$. Then the left-hand side of (10.9) is equal to $E_{\mathcal{N}}(\beta)$ and hence

$$E_{\mathcal{N}}(\beta)\geqslant-\frac{1}{2}\left(\int_{-\infty}^{\infty}dx\;|Q(x)|\right)^2-\frac{\beta^2}{2}\left(\int_{\{|P(x)|>M\}}dx\;|P(x)|\right)^2-\beta M.\tag{10.10}$$

Since by choosing M large enough we can make the second term on the right-hand side of (10.10) as small as we please, it follows that $E_{\mathcal{N}}(\beta)=o(\beta^2)$ as $\beta\to+\infty$. Thus $E_{\mathcal{N}}(\beta)>-\beta^2$ for β sufficiently large, while $E_{\mathcal{N}}(0)=-\kappa_{\mathcal{N}}^2<0$. Hence by the intermediate value theorem, the equation $E_{\mathcal{N}}(\beta)=-\beta^2$ has at least one solution. A similar argument shows that each of the remaining eigenvalue branches $E_j(\beta)$ for $j=1,\ldots,\mathcal{N}-1$ must intersect the parabola $E=-\beta^2$ at least once. Since each intersection increases the number of negative-energy bound states of (1.1), the proof of (i) is complete. Note that if an eigenvalue branch $E_j(\beta)$ touches or intersects the parabola $E=-\beta^2$ at other points, such additional points are also responsible for additional negative-energy bound states of (1.1). Moreover, there may be other eigenvalue branches $E(\beta)$ starting at $(\tilde{\beta},0)$ for some $\tilde{\beta}>0$ and intersecting or touching the parabola $E=-\beta^2$ at one or more points; again, each of such points also increases the number of negative-energy bound states of (1.1).

(ii) If $P(x)\equiv 0$, each eigenvalue branch $E_0(\beta)$ becomes the horizontal line $E_0(\beta)=-\beta_0^2$ for $\beta\geq 0$, and hence $E_0''(\beta)=0$ for $\beta>0$. Thus in the rest of the analysis we can assume that $P(x)\not\equiv 0$. Associated with the eigenvalue $E_0(\beta)$ there exists[25] a real-valued, analytic eigenvector $\psi(\beta, x)$. Near $\beta=\beta_0$ we have the convergent expansions

$$E_0(\beta)=\sum_{n=0}^{\infty} a_n(\beta-\beta_0)^n, \quad \psi(\beta, x)=\sum_{n=0}^{\infty} \psi_n(x)(\beta-\beta_0)^n, \tag{10.11}$$

with $\psi_n\in L^2(\mathbf{R})$ for $n\geq 0$. Substituting (10.11) in (1.1) we get the following set of equations (see pp. 333 and 334 of Ref. 26) for $n\geq 0$:

$$\psi_n''(x)-\beta_0^2\psi_n(x)+a_1\psi_{n-1}(x)+a_2\psi_{n-2}(x)=[-\beta_0P(x)+Q(x)]\psi_n(x)-P(x)\psi_n(x) -\sum_{j=3}^{n} a_j\psi_{n-j}(x), \tag{10.12}$$

where it is assumed that $a_{-n}=\psi_{-n}(x)=0$ if $n\geq 1$. From (9.17) and (9.18) we see that

$$a_0=E_0(\beta_0), \quad a_1=-\frac{1}{\|\psi_0\|_2^2}\int_{-\infty}^{\infty}dx\ P(x)\psi_0(x)^2. \tag{10.13}$$

We may choose $\psi_0(x)=f_l^+(i\beta_0,x)$. It suffices to prove that $1/T^+(i\beta)$ has a zero of order at least m at β_0 if and only if $E_0(\beta)+\beta^2$ has a zero of order at least m at β_0. From Proposition 8.1 we know that this is true when $m=1$. If β_0 is a zero of $E_0(\beta)+\beta^2$ of order m for some $m\geq 2$, then the coefficients a_n in (10.3) are determined for $n=0,1,\ldots,m-1$ by expanding $E_0(\beta)+\beta^2$ about β_0. Thus for $m=2$ we get $a_0=-\beta_0^2$, $a_1=-2\beta_0$; for $m=3$ we get $a_0=-\beta_0^2$, $a_1=-2\beta_0$, $a_2=-1$; for $m\geq 4$ we get $a_0=-\beta_0^2$, $a_1=-2\beta_0$, $a_2=-1$, and $a_3=\cdots=a_{m-1}=0$. Then, comparing (10.12) and (8.2) and using the fact that the functions $g_{l,n}^+(i\beta_0,x)$ are uniquely determined as solutions of (8.2) by the requirement that $g_{l,n}^+(i\beta_0,\cdot)\in L^2(\mathbf{R})$, we obtain

$$\psi_n(x)=i^n g_{l,n}^+(i\beta_0,x), \quad n=0,\ldots,m-1. \tag{10.14}$$

Thus, by Theorem 8.4, we see that $1/T^+(i\beta)$ has a zero of order at least m at β_0. Conversely, suppose $1/T^+(i\beta)$ has a zero of order at least m at β_0. From (10.12) one can derive (see p. 334 of Ref. 26) the following recursion formula for the coefficients a_n:

$$a_n=-\frac{1}{\|\psi_0\|_2^2}\int_{-\infty}^{\infty}dx\ \psi_0(x)\left(P(x)\psi_{n-1}(x)+\sum_{j=1}^{n-1} a_j\psi_{n-j}(x)\right), \quad n\geq 2. \tag{10.15}$$

Now assume $m\geq 2$. Since the functions $\{g_{l,n}^+(i\beta_0,\cdot)\}_{n=0}^{m-1}$ form a Jordan chain of length m, using (8.13) with $n=1$ and (10.13) we get $a_1=-2\beta_0$. Hence $E_0(\beta)+\beta^2$ has a zero of order at least 2 at β_0. If $m\geq 3$, then using (8.13) with $n=2$, (10.14), and (10.15) we obtain $a_2=-1$ and this, in turn, implies that $E_0(\beta)+\beta^2$ has a zero of order at least 3 at β_0. If $m\geq 4$, then (8.13) and (10.15) give $a_3=0$ and then $a_j=0$ for all $j=3,\ldots,m-1$. As a result, $E_0(\beta)+\beta^2$ has a zero of order at least m at β_0. If m is even, then $E_0(\beta)$ touches the parabola $E=-\beta^2$ at β_0 but stays either above or below the parabola; if m is odd, then $E_0(\beta)$ intersects the parabola by crossing from one side to the other.

(iii) For the lowest eigenvalue, we need to show that a_2 in (10.11) is negative for any $\beta_0>0$. We have (see p. 334 of Ref. 26)

$$\psi_1(x)=-\psi_0(x)\int_{x_0}^{x}\frac{dt}{\psi_0(t)^2}\int_{-\infty}^{t}ds\;\psi_0(s)[P(s)\psi_0(s)+a_1\psi_0(s)], \tag{10.16}$$

where the constant x_0 is arbitrary; however, since changing x_0 amounts to adding a constant multiple of $\psi_0(x)$ to $\psi_1(x)$, with the help of (10.6) one can show that the value of a_n given in (10.15) is independent of x_0. Using (10.16) in (10.15) with $n=2$ and the positivity of $\psi_0(x)$, after performing an integration by parts, we get

$$a_2=-\frac{1}{\|\psi_0\|_2^2}\int_{-\infty}^{\infty}\frac{dx}{\psi_0(x)^2}\left(\int_{-\infty}^{x}dt\;\psi_0(t)^2[P(t)+a_1]\right)^2<0.$$

Thus $a_2<0$ in (10.11) for any $\beta_0>0$, and hence we have $E''_{\mathcal{N}}(\beta)<0$ for any $\beta>0$.

(iv) Let us consider the number of zeros of $f_l^+(i\beta,x)$ in relation to the behavior of the eigenvalue branch $E_0(\beta)$ near β_0. From Proposition 10.3 (i), when $Q(x)$ in (2.6) is replaced by $Q(x)-\beta P(x)$, we know that the number of zeros of $f_l^+(i\beta,x)$ is equal to the number of eigenvalue branches lying below $-\beta^2$. Let I_{β_0} denote the interval $(\beta_0-\epsilon,\beta_0+\epsilon)$ and let J_{β_0} denote $(\beta_0-\epsilon,\beta_0)\cup(\beta_0,\beta_0+\epsilon)$ for sufficiently small $\epsilon>0$, and let us consider the number of eigenvalue branches below $-\beta^2$ when $\beta\in I_{\beta_0}$. If m is even, then $E_0(\beta)$ touches the parabola $E=-\beta^2$ at β_0 but stays either above or below that parabola; in the former case $E_0(\beta)>-\beta^2$ for $\beta\in J_{\beta_0}$ and hence the number of zeros of $f_l^+(i\beta,x)$ remains unchanged for $\beta\in I_{\beta_0}$; in the latter case $E_0(\beta)<-\beta^2$ for $\beta\in J_{\beta_0}$ and hence the number of zeros of $f_l^+(i\beta,x)$ for $\beta\in J_{\beta_0}$ is exactly one more than the number of zeros of $f_l^+(i\beta_0,x)$. If m is odd, then $E_0(\beta)$ intersects the parabola $E=-\beta^2$ by crossing from one side to the other of that parabola; if $E_0(\beta)<-\beta^2$ on $(\beta_0-\epsilon,\beta_0)$, then the number of zeros of $f_l^+(i\beta,x)$ decreases by one as β increases through β_0; if $E_0(\beta)>-\beta^2$ on $(\beta_0-\epsilon,\beta_0)$, then the number of zeros increases by one as β increases through β_0. In order to prove that the number of zeros of $f_l^+(i\beta,x)$ can only change if β corresponds to a bound state of (1.1) with real (negative) energy, we can proceed as follows. If β_1 and β_2 with $\beta_1<\beta_2$ correspond to two consecutive real bound-state energies of (1.1), then no eigenvalue branch can intersect the parabola $E=-\beta^2$ for $\beta\in(\beta_1,\beta_2)$. Hence the number of eigenvalue branches that lie below $-\beta^2$ is constant for $\beta\in(\beta_1,\beta_2)$, or equivalently, the number of zeros of $f_l^+(i\beta,\ x)$ is constant for $\beta\in(\beta_1,\beta_2)$. ■

We remark that statement (iii) is a familiar result that also follows from the min-max principle. An example illustrating Theorem 10.6 is given in the next section (Example 11.3).

Proposition 10.7: Assume $P,Q\in L^1(\mathbf{R})$ and let $k_0=\alpha+i\beta$ for some $\alpha\neq 0$ and $\beta>0$. If $P(x)\leqslant 2\beta$, then $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ cannot vanish for any $x\in\mathbf{R}$.

Proof: Using (4.7) in (4.16) we obtain

$$\frac{d}{dx}[f_l^+(-\overline{k_0},x);f_l^+(k_0,x)]=2i\alpha[P(x)-2\beta]|f_l^+(k_0,x)|^2. \tag{10.17}$$

Suppose $f_l^+(k_0,x)$ has at least one zero and let d be the right-most zero of $f_l^+(k_0,x)$. Note that, as seen from (4.7), the zeros of $f_l^+(-\overline{k_0},x)$ and $f_l^+(k_0,x)$ coincide. Integrating (10.17) over $(d,+\infty)$ and using (8.1) and (8.2), we obtain

$$2i\alpha\int_d^{\infty}dx\;[P(x)-2\beta]|f_l^+(k_0,x)|^2=0.$$

This is impossible if $\alpha\neq 0$ and $P(x)\leqslant 2\beta$; note that $P(x)=2\beta$ on a semi-infinite interval would contradict $P\in L^1(\mathbf{R})$. Hence $f_l^+(k_0,x)$ cannot vanish for any $x\in\mathbf{R}$. The proof for $f_r^+(k_0,x)$ is analogous. ■

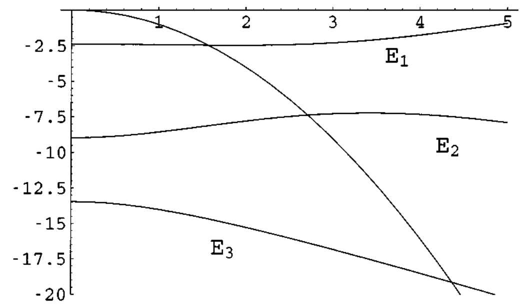

FIG. 1. Eigenvalue curves $E_1(\beta)$, $E_2(\beta)$, $E_3(\beta)$ intersecting the parabola $E=-\beta^2$.

XI. EXAMPLES

In this section we illustrate our results on the number and location of the zeros of $1/T^+(k)$ and on the zeros of the Jost solutions with some explicit examples. Example 11.1 shows the typical behavior of some eigenvalues branches as it was discussed in Secs. IX and X. Example 11.2 exhibits the possibility of real zeros of $1/T^+(k)$ and of complex zeros off the imaginary axis in $\mathbf{C}^+$. In Example 11.3 we consider the zeros on the imaginary axis and the corresponding eigenvalue branches. We also demonstrate the possibility of a double zero on the positive imaginary axis. While Examples 11.1–11.3 involve simple step potentials, Example 11.4 concerns potentials that decay exponentially.

Example 11.1: For real parameters $a_\pm$ and $b_\pm$, let

$$P(x)=\begin{cases} b_+, & x\in(0,1),\\ b_-, & x\in(-1,0),\\ 0, & \text{elsewhere,}\end{cases}\qquad Q(x)=\begin{cases} a_+, & x\in(0,1),\\ a_-, & x\in(-1,0),\\ 0, & \text{elsewhere.}\end{cases}\tag{11.1}$$

Then by straightforward calculations one obtains

$$\frac{e^{-2ik}}{T^+(k)}=\cos s_+\cos s_-+F_+\sin s_+\cos s_-+F_-\cos s_+\sin s_--G\sin s_+\sin s_-,\tag{11.2}$$

where

$$s_\pm=\sqrt{k^2-ikb_\pm-a_\pm},\quad F_\pm=\frac{k^2+s_\pm^2}{2iks_\pm},\quad G=\frac{s_+^2+s_-^2}{2s_+s_-}.$$

For $a_\pm=-15$, $b_+=2$, and $b_-=-2$ we get three purely imaginary bound states at $k=1.56\overline{8}i$, $k=2.71\overline{7}i$, and $k=4.37\overline{6}i$. Here and below we use an overline on the last digit to indicate round-off. These are all the bound states because a plot of arg $T^+(k)$ for $k\in\mathbf{R}^+$ reveals that arg $T^+(0+)=5\pi/2$, so that by (9.22) we have $N(P,Q)=3$; note that we are in the generic case. Figure 1 shows the eigenvalue curves associated with the potential $V(\beta,x)=Q(x)-\beta P(x)$ [cf. (9.16)]. The analysis shows that the branches $E_2(\beta)$ and $E_1(\beta)$ are not concave, only $E_3(\beta)$ is concave down. This, in particular, illustrates part (iii) of Theorem 10.6.

Example 11.2: In this example we demonstrate the existence of nonreal zeros of $1/T^+(k)$. Putting $a_-=0$, $a_+=a$, $b_-=0$, and $b_+=b$ in (11.1) and (11.2), we get

$$\frac{1}{T^+(k)}=e^{ik}\left[\cos s+\frac{k^2+s^2}{2iks}\sin s\right],\tag{11.3}$$

where $s=\sqrt{k^2-ibk-a}$. By Theorem 9.3, if $a=0$ and $b>2$, then we must have a bound state at $k=i\beta$ for some positive β. Indeed, if $a=0$ and $b=21/10$, we obtain a bound state at $k=0.1\overline{5}i$ and this is the only bound state. When $a=-9$ and $b=5$, we find bound states at $k_1=5.61\overline{9}i$, $k_2^\pm$

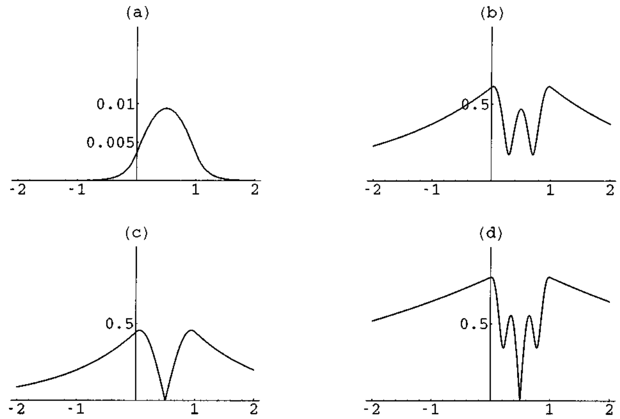

FIG. 2. (a) $f_l^+(k_1,x)$, (b) $|f_l^+(k_2^+,x)|$, (c) $|f_l^+(k_3^+,x)|$, and (d) $|f_l^+(k_4^+,x)|$.

$=\pm 6.28+0.495i$, $k_3^\pm=\pm 0.838+0.81i$, and $k_4^\pm=\pm 9.898+0.2192i$. As in Example 11.1 it follows that these are all the bound states. In Fig. 2 we plot $f_l^+(k_1,x)$, $|f_l^+(k_2^+,x)|$, $|f_l^+(k_3^+,x)|$, and $|f_l^+(k_4^+,x)|$ on the interval $x\in(-2,2)$. Let us remark that the functions $f_l^+(k_1,x)$ and $f_l^+(k_2^\pm,x)$ are even with respect to $x=1/2$, and the functions $f_l^+(k_3^\pm,x)$ and $f_l^+(k_4^\pm,x)$ are odd with respect to $x=1/2$. When $a=-9.2738$ and $b=3.9708$, we obtain a zero of $1/T^+(k)$ at $k=1$. In the special case when $a=0$ we have

$$\frac{1}{T^+(0)}=1-\frac{b}{2},\quad \frac{L^+(0)}{T^+(0)}=\frac{R^+(0)}{T^+(0)}=\frac{b}{2}.$$

Hence $1/T^+(0)=0$ if and only if $b=2$. Note that the bound states may occur even when $Q(x)\geqslant 0$. For example, when $a=1$ and $b=10$, we obtain over 200 bound states, four of which correspond to k values on the positive imaginary axis with $k=i\beta_j$, where

$$\beta_1=0.13,\quad \beta_2=2.50,\quad \beta_3=5.63,\quad \beta_4=9.16.$$

When $a=0$ and $b=100$, we obtain 31 bound states on the positive imaginary axis with $k=i\beta_j$, where

$$\beta_1=0.10,\quad \beta_2=0.41,\quad \beta_3=0.93,\quad \beta_4=1.67,\quad \beta_5=2.64,\quad \beta_6=3.85,\quad \beta_7=5.33,$$

$$\beta_8=7.09,\quad \beta_9=9.19,\quad \beta_{10}=11.69,\quad \beta_{11}=14.63,\quad \beta_{12}=18.20,\quad \beta_{13}=22.61,$$

$$\beta_{14}=28.43,\quad \beta_{15}=37.63,\quad \beta_{16}=60.41,\quad \beta_{17}=69.69,\quad \beta_{18}=75.60,\quad \beta_{19}=80.11,$$

$$\beta_{20}=83.77,\quad \beta_{21}=86.83,\quad \beta_{22}=89.42,\quad \beta_{23}=91.63,\quad \beta_{24}=93.52,\quad \beta_{25}=95.12,$$

$$\beta_{26}=96.46,\quad \beta_{27}=97.57,\quad \beta_{28}=98.46,\quad \beta_{29}=99.14,\quad \beta_{30}=99.62,\quad \text{and}\quad \beta_{31}=99.91,$$

and there are also many more bound states corresponding to k-values off the imaginary axis in $\mathbf{C}^+$. In this case, one finds that $f_l^+(i\beta,x)$ has no zeros for $\beta=0$, no zeros for $\beta\in(\beta_{31},+\infty)$, one zero for $\beta\in(0,\beta_1)$, and one zero for $\beta\in(\beta_{30},\beta_{31})$, j zeros for $\beta\in(\beta_{j-1},\beta_j)$ and j zeros for $\beta\in(\beta_{31-j},\beta_{32-j})$ with $j=2,3,...,15$, and 16 zeros for $\beta\in(\beta_{16},\beta_{17})$.

Example 11.3: (a) Let $a=0$ and $b=10$ in Example 11.2, and hence $Q(x)=0$ and $P(x)\geqslant 0$. Note that $N(0,Q)=0$. From (9.16) we obtain [cf. (11.3)]

$$2\Delta\sqrt{-E}\,\cos\Delta=(\Delta^2+E)\sin\Delta, \tag{11.4}$$

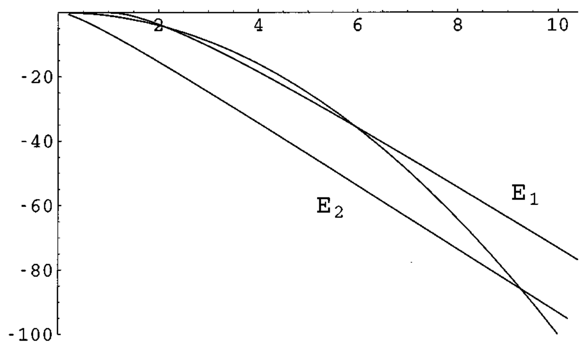

FIG. 3. The parabola $E=-\beta^2$ intersecting the eigenvalue curves $E_1(\beta)$ and $E_2(\beta)$.

where $E=E(\beta)$ is the energy in (9.16) and $\Delta=\sqrt{\beta b-a+E}$. Using the half-angle formula for the tangent function, we can write (11.4) as a pair of equations determining the eigenvalue curves:

$$\tan\left(\frac{\Delta}{2}\right)=\frac{\sqrt{-E}}{\Delta}, \quad \tan\left(\frac{\Delta}{2}\right)=-\frac{\Delta}{\sqrt{-E}}. \tag{11.5}$$

Recall from the proof of Theorem 9.7 that the bound states of (1.1) with real energies correspond to the β-values where the eigenvalue curves intersect the parabola $E=-\beta^2$. When $a=0$ and $b=10$, from (11.5) we obtain two eigenvalue branches intersecting the parabola $E=-\beta^2$. Let $E_2(\beta)$ denote the eigenvalue branch responsible for the lowest real bound-state energy. We see that $E_2(\beta)$ emerges from (0,0) and intersects the parabola $E=-\beta^2$ at $\beta_3=9.273$. The second eigenvalue branch, $E_1(\beta)$, emerges from zero at $\beta=\pi^2/10$ and then intersects the parabola $E=-\beta^2$ at $\beta_1=2.114$ and at $\beta_2=5.963$. These eigenvalue branches and the parabola $E=-\beta^2$ are plotted in Fig. 3. The values β_1, β_2, and β_3 correspond to simple zeros of $1/T^+(i\beta)$. If $\beta\geqslant\beta_3$, then $f_l^+(i\beta,x)$ has no zeros. If $\beta\in[\beta_2,\beta_3)$, then $f_l^+(i\beta,x)$ has one zero because $E_2(\beta)$ is the only eigenvalue below $-\beta^2$. If $\beta\in(\beta_1,\beta_2)$, then $f_l^+(i\beta,x)$ has two zeros because both $E_2(\beta)$ and $E_1(\beta)$ lie below $-\beta^2$. If $\beta\in(0,\beta_1]$ then $f_l^+(i\beta,x)$ has one zero, and if $\beta=0$ then $f_l^+(0,x)$ has no zeros because (2.6) with $Q(x)=0$ has no bound states.

(b) When $a=0$, one can choose the parameter b such that the branch $E_1(\beta)$ just touches the parabola $E=-\beta^2$ at β_1. Then the slope of the eigenvalue curve at β_1 must be equal to $-2\beta_1$, and this happens when

$$\tan\left(\frac{\sqrt{b^2-4}}{4}\right)=-\frac{\sqrt{b+2}}{\sqrt{b-2}},$$

from which we get $b=9.2066$, leading to $\beta_1=3.603$, and β_1 corresponds to a double zero of $1/T^+(i\beta)$. The eigenvalue branch $E_2(\beta)$ intersects the parabola $E=-\beta^2$ at β_2; we have $\beta_2=8.433$, which is responsible for the lowest real bound-state energy. In this case, $f_l^+(i\beta,\, x)$ has no zeros for $\beta=0$, one zero for $\beta\in(0,\beta_2)$, and no zeros for $\beta\in[\beta_2,+\infty)$. We show the two eigenvalue branches and the parabola $E=-\beta^2$ in Fig. 4.

(c) When $b=10$ we can find a such that the lowest real bound-state energy corresponds to a double zero of $1/T^+(i\beta)$. Proceeding as in (b), we get

$$a=\frac{10\beta_1(\beta_1-4)}{\beta_1-3},$$

where β_1 is obtained by solving

$$\tan\left(\frac{\sqrt{10\beta+3\beta^2-\beta^3}}{2\sqrt{\beta-3}}\right)=\frac{\beta\sqrt{\beta-3}}{\sqrt{10\beta+3\beta^2-\beta^3}}. \tag{11.6}$$

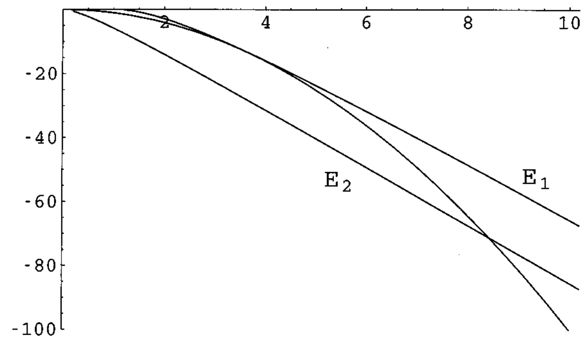

FIG. 4. The parabola $E=-\beta^2$ touching $E_1(\beta)$ and intersecting $E_2(\beta)$.

From (11.6) we get $\beta_1=4.72\overline{4}$ and hence $a=19.85\overline{2}$. In this case $f_l^+(i\beta,x)$ has no zeros for any $\beta\geqslant 0$. The eigenvalue curve $E_1(\beta)$ and the parabola $E=-\beta^2$ are plotted in Fig. 5, and it is seen that there are no other real bound-state energies besides $-\beta_1^2$.

Example 11.4: Let

$$\eta_l^+(k,x)=1-\frac{2i(1+b)\epsilon}{k+i\epsilon}\frac{ce^{-2\epsilon x}}{1+ce^{-2\epsilon x}},\quad x\geqslant 0, \tag{11.7}$$

$$\eta_r^+(k,x)=1-\frac{2i(1+b)\epsilon}{k+i\epsilon}\frac{ce^{2\epsilon x}}{1+ce^{2\epsilon x}},\quad x\geqslant 0, \tag{11.8}$$

where c, ϵ are positive parameters and b is a real parameter. Using (1.1), (1.2), (3.3), (3.4), (11.7), and (11.8), we obtain

$$P(x)=\frac{4b\epsilon ce^{-2\epsilon|x|}}{1+ce^{-2\epsilon|x|}},$$

$$Q(x)=\frac{4\epsilon^2ce^{-2\epsilon|x|}[-3b-2+b^2ce^{-2\epsilon|x|}]}{(1+ce^{-2\epsilon|x|})^2}.$$

Hence $e^p=(1+c)^{2b}$ and using (6.3) we get

$$T^+(k)=\frac{k(k+i\epsilon)^2e^p}{(k-k_0)(k-k_+)(k-k_-)},$$

where we have defined

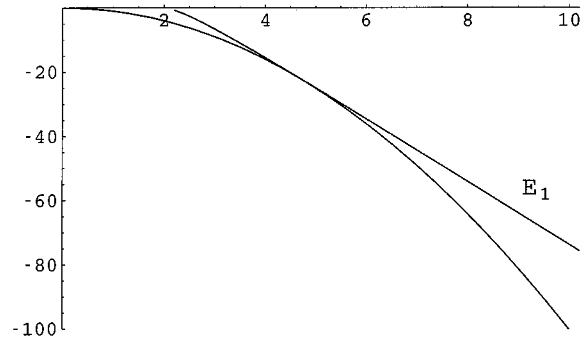

FIG. 5. The parabola $E=-\beta^2$ touching the eigenvalue curve $E_1(\beta)$.

$$k_0 = i\,\frac{\epsilon}{1+c}\,[-1+c+2bc],$$

$$k_{\pm} = \frac{i\epsilon}{2(1+c)}\,[(-1+c+4bc) \pm \sqrt{1+c^2+14c+16bc}]. \tag{11.9}$$

Let us now analyze the poles of $T^+(k)$. First note that ϵ simply acts as a scaling parameter for the location of the poles; thus the relevant parameters are b and c. We can divide the half-plane $\{(c,b):c>0\}$ into four separate regions by using the three nonintersecting curves $b=\Gamma_+(c)$, $b=\Gamma_0(c)$, and $b=\Gamma_-(c)$, where

$$\Gamma_0(c) = \frac{1-c}{2c}, \quad \Gamma_{\pm}(c) = -\frac{c-3}{4c} \pm \sqrt{\frac{(c-3)^2}{16c^2} + \frac{1}{c}}.$$

On these three curves the exceptional case occurs; note that $k_-=0$ on Γ_+, $k_+=0$ on Γ_-, and $k_0=0$ on Γ_0. The number of bound states changes by one as we cross each of these three curves; otherwise, we are in the generic case. Note that:

(i) If $b>\Gamma_+(c)$, then k_0, k_+, and k_- all lie on the positive imaginary axis, and hence we have three bound states.

(ii) If $\Gamma_0(c)<b\leqslant\Gamma_+(c)$, then k_0 and k_+ lie on the positive imaginary axis, but $k_- \notin \mathbf{C}^+$; hence we have two bound states.

(iii) If $\Gamma_-(c)<b\leqslant\Gamma_0(c)$, then there is exactly one bound state because k_+ lies on the positive imaginary axis but k_0 and k_- are not in $\mathbf{C}^+$.

(iv) There are no bound states when $b\leqslant\Gamma_-(c)$ because none of k_0, k_+, and k_- lie in $\mathbf{C}^+$. In this case, k_0 is always located on the imaginary axis; k_+ and k_- lie on the imaginary axis when $b\geqslant -(c+14+1/c)/16$ and they lie in $\mathbf{C}^-$ symmetrically located with respect to the imaginary axis.

This example can also be specialized to show the occurrence of a double zero of $1/T^+(k)$. Indeed, choose $b=-(c+14+1/c)/16$ and $c\in(-1,-5+2\sqrt{5})$. Then, from (11.9) we see that $k_+=k_-=-i\epsilon(c^2+10c+5)/[8(1+c)]$, and hence $T^+(k)$ has a double pole on the positive imaginary axis in $\mathbf{C}^+$ for any $c\in(-1,-5+2\sqrt{5})$. Note also that when $b=(1-c)/(4c)$ and $c\in(-1,-5+2\sqrt{5})$, although k_0 is located on the negative imaginary axis, k_+ and k_- are symmetrically located on the real axis; thus, in this case $T^+(k)$ has poles on the real axis. When $b=-(5+\sqrt{5})/10$ and $c=-5+2\sqrt{5}$, both k_+ and k_- vanish, and hence we get a simple pole for $T^+(k)$ at $k=0$; this illustrates the exceptional case when the denominator in (5.3) vanishes.

APPENDIX: SMALL-*k* ESTIMATES

In this Appendix, proceeding as in Refs. 11 and 12, we obtain various small-k estimates that are needed in the proof of Theorem 5.2.

In the exceptional case, let $\tilde{\psi}(k,x)$ be the solution of (1.1) satisfying the initial conditions

$$\tilde{\psi}(k,0) = f_l(0,0), \quad \tilde{\psi}'(k,0) = f_l'(0,0), \quad k\in\mathbf{R}. \tag{A1}$$

Note that $\tilde{\psi}(0,x)=f_l(0,x)$, and hence $\tilde{\psi}(0,x)$ is bounded in such a way that $\tilde{\psi}(0,+\infty)=1$ and $\tilde{\psi}(0,-\infty)=\gamma$, where γ is the constant defined in (2.9). We have

$$\tilde{\psi}(k,x) = f_l(0,0)\cos kx + f_l'(0,0)\,\frac{\sin kx}{k} + \frac{1}{k}\int_0^x dy\,\sin k(x-y)[ikP(y)+Q(y)]\tilde{\psi}(k,y). \tag{A2}$$

Let $\psi_1(k,x)$ denote the solution of (1.1) with $P(x)=0$ and satisfying (A1). We have

$$\psi_1(k,x) = f_l(0,0)\cos kx + f_l'(0,0)\,\frac{\sin kx}{k} + \frac{1}{k}\int_0^x dy\,\sin k(x-y)Q(y)\psi_1(k,y). \tag{A3}$$

Note that $\tilde{\psi}(0,x)=\psi_1(0,x)$.

Proposition A.1: Assume $Q\in L_1^1(\mathbf{R})$. For $x,k\in\mathbf{R}$, we have

$$|\psi_1(k,x)-\psi_1(0,x)|\leqslant C\left(\frac{|kx|}{1+|kx|}\right)^2, \quad |\psi_1(k,x)|\leqslant C(1+|k|). \tag{A4}$$

Proof: Note that $\psi_1(0,x)=f_l(0,x)$ and hence it is uniformly bounded for $x\in\mathbf{R}$. Furthermore,

$$\psi_1(0,x)=f_l(0,0)+xf_l'(0,0)+\int_0^x dy\ (x-y)Q(y)\psi_1(0,y). \tag{A5}$$

Subtracting (A5) from (A3) and iterating the resulting integral equation as in the proof of Proposition A.1 of Ref. 12 (using $H(x)\equiv H_+=1$ there) we obtain the first inequality in (A4). Using that inequality and the boundedness of $\psi_1(0,x)$, we obtain the second inequality in (A4). ■

Let us choose a second linearly independent solution of (1.1) with $P(x)=0$ such that the Wronskian $[\psi_1(k,x);\psi_2(k,x)]$ is equal to 1. For example, we can choose $\psi_2(k,x)$ so that it satisfies the initial conditions $\psi_2(k,0)=0$ and $\psi_2'(k,0)=1/f_l(0,0)$; note that there is no loss of generality in assuming $f_l(0,0)\neq 0$, since the case $f_l(0,0)=0$ can be handled by a shift of the origin. We have

$$\psi_2(k,x)=\frac{\sin kx}{kf_l(0,0)}+\frac{1}{k}\int_0^x dy\ \sin k(x-y)Q(y)\psi_2(k,y). \tag{A6}$$

Proposition A.2: Assume $Q\in L_1^1(\mathbf{R})$. Then, for $x,k\in\mathbf{R}$ we have

$$|\psi_2(k,x)|\leqslant\frac{C|x|}{1+|kx|}, \quad |\psi_2(k,x)-\psi_2(0,x)|\leqslant C|x|\left(\frac{|kx|}{1+|kx|}\right)^2. \tag{A7}$$

Proof: Iterating (A6) as in the proof of Proposition A.1 of Ref. 12, we obtain the first inequality in (A7). Note that from (A6) we have

$$\psi_2(0,x)=\frac{x}{f_l(0,0)}+\int_0^x dy\ (x-y)Q(y)\psi_2(0,y). \tag{A8}$$

Subtracting (A8) from (A6) and iterating the resulting integral equation, we obtain the second inequality in (A7). ■

Proposition A.3: Assume $P,Q\in L_1^1(\mathbf{R})$. Then, for $x\in\mathbf{R}$ and as $k\to 0$ in $\mathbf{R}$, we have

$$\tilde{\psi}(k,x)-\psi_1(k,x)=-ik\psi_1(0,x)\int_0^x dz\ \psi_2(0,z)P(z)\psi_1(0,z)$$
$$+ik\psi_2(0,x)\int_0^x dz\ P(z)\psi_1(0,z)^2+O\left(|kx|\left[\frac{|kx|}{1+|kx|}\right]^2\right). \tag{A9}$$

Proof: Recall that $\psi_1(k,x)$ and $\psi_2(k,x)$ are two linearly independent solutions of (1.1) when $P(x)=0$. Using variation of parameters on (1.1), we obtain (assuming $x\geqslant 0$)

$$\tilde{\psi}(k,x)-\psi_1(k,x)=-ik\psi_1(k,x)\int_0^x dz\ \psi_2(k,z)P(z)\tilde{\psi}(k,z)$$
$$+ik\psi_2(k,x)\int_0^x dz\ \psi_1(k,z)P(z)\tilde{\psi}(k,z). \tag{A10}$$

Let us write (A10) as

$$\tilde{\psi}(k,x)-\psi_1(k,x)=A_1+A_2+A_3+A_4+A_5+A_6+A_7+B_1+B_2, \tag{A11}$$

where we have defined

$$B_1 = -ik\psi_1(k,x)\int_0^x dz\ \psi_2(k,z)P(z)[\tilde{\psi}(k,z)-\psi_1(k,z)],$$

$$B_2 = ik\psi_2(k,x)\int_0^x dz\ \psi_1(k,z)P(z)[\tilde{\psi}(k,z)-\psi_1(k,z)],$$

$$A_1 = -ik\psi_1(0,x)\int_0^x dz\ \psi_2(0,z)P(z)\psi_1(0,z),$$

$$A_2 = -ik[\psi_1(k,x)-\psi_1(0,x)]\int_0^x dz\ \psi_2(k,z)P(z)\psi_1(k,z),$$

$$A_3 = -ik\psi_1(0,x)\int_0^x dz\ [\psi_2(k,z)-\psi_2(0,z)]P(z)\psi_1(k,z),$$

$$A_4 = -ik\psi_1(0,x)\int_0^x dz\ \psi_2(0,z)P(z)[\psi_1(k,z)-\psi_1(0,z)],$$

$$A_5 = ik\psi_2(0,x)\int_0^x dz\ P(z)\psi_1(0,z)^2,$$

$$A_6 = ik[\psi_2(k,x)-\psi_2(0,x)]\int_0^x dz\ P(z)\psi_1(k,z)^2,$$

$$A_7 = ik\psi_2(0,x)\int_0^x dz\ [\psi_1(k,z)-\psi_1(0,z)][\psi_1(k,z)+\psi_1(0,z)]P(z).$$

Using the estimates in (A4) and (A7), we obtain

$$|A_2| \le C|k|(1+|k|)\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^x dz\ z|P(z)|, \tag{A12}$$

$$|A_3| \le C|k|(1+|k|)\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^x dz\ z|P(z)|, \tag{A13}$$

$$|A_4| \le C|k|\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^x dz\ z|P(z)|, \tag{A14}$$

$$|A_6| \le C|kx|(1+|k|)^2\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^x dz\ |P(z)|, \tag{A15}$$

$$|A_7| \le C|kx|(1+|k|)\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^x dz\ |P(z)|. \tag{A16}$$

For $x<0$ analogous estimates hold. Iterating the integral equation for $\tilde{\psi}(k,x)-\psi_1(k,x)$ given in (A11) and using (A12)–(A16), we obtain (A9). ■

In order to estimate the small-k asymptotics of $T^+(k)$, we will use (4.2). Note that as in (A24) of Ref. 12 we have

$$f_l(0,0)[f_l^+(k,x);f_r^+(k,x)]=f_r^+(k,0)\left[-ikf_l(0,0)+f_l'(0,0)+\int_0^\infty dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right]$$
$$-f_l^+(k,0)\left[ikf_l(0,0)+f_l'(0,0)-\int_{-\infty}^0 dz\ e^{-ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right]. \tag{A17}$$

Proposition A.4: Assume $P,Q\in L_1^1(\mathbf{R})$. Then, as $k\to 0$ we have

$$\int_0^\infty dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)=-f_l'(0,0)+ikf_l(0,0)-ik+ik\int_0^\infty dz\ P(z)f_l(0,z)^2+o(|k|), \tag{A18}$$

$$\int_{-\infty}^0 dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)=f_l'(0,0)-ik\gamma+ikf_l(0,0)+(ik/\gamma)\int_{-\infty}^0 dz\ P(z)f_l(0,z)^2+o(|k|), \tag{A19}$$

where γ is the constant defined in (2.9).

Proof: Let us write

$$\int_0^\infty dz\ e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)=J_1+J_2+J_3+J_4+J_5+J_6, \tag{A20}$$

where

$$J_1=\int_0^\infty dz\ Q(z)\tilde{\psi}(0,z),\quad J_2=\int_0^\infty dz\ Q(z)[e^{ikz}-1]\tilde{\psi}(0,z),$$

$$J_3=ik\int_0^\infty dz\ e^{ikz}P(z)\tilde{\psi}(0,z),$$

$$J_4=\int_0^\infty dz\ e^{ikz}Q(z)[\psi_1(k,z)-\tilde{\psi}(0,z)],$$

$$J_5=ik\int_0^\infty dz\ e^{ikz}P(z)[\tilde{\psi}(k,z)-\tilde{\psi}(0,z)],$$

$$J_6=\int_0^\infty dz\ e^{ikz}Q(z)[\tilde{\psi}(k,z)-\psi_1(k,z)].$$

As in (A25) and (A26) of Ref. 12 we have $J_1=-f_l'(0,0)$ and

$$J_2=ik[f_l(0,0)-1]+o(|k|),\quad k\to 0.$$

As $k\to 0$, using (A4) we obtain $J_4=o(|k|)$ and

$$J_3=ik\int_0^\infty dz\ P(z)\tilde{\psi}(0,z)+o(|k|),$$

and using (A9) we have $J_5=o(|k|)$ and

$$J_6=ik\int_0^\infty dz\ Q(z)\left[-\psi_1(0,z)\int_0^z dt\ \psi_2(0,t)P(t)\psi_1(0,t)+\psi_2(0,z)\int_0^z dt\ P(t)\psi_1(0,t)^2\right]+o(|k|). \tag{A21}$$

Notice that $[\psi_1(k,x);\psi_2(k,x)]=1$ and $Q(z)\psi_s(0,z)=\psi_s''(0,z)$ for $s=1,2$. Therefore, using $\psi_2'(0,+\infty)=1$, $\psi_1'(0,z)=o(z^{-1})$ as $z\to+\infty$, $\psi_1(0,z)=\tilde{\psi}(0,z)$, and integration by parts twice in (A21), we obtain

$$J_6 = -ik \int_0^\infty dz\ P(z)\overline{\psi}(0,z) + ik \int_0^\infty dz\ P(z)\overline{\psi}(0,z)^2 + o(|k|), \quad k \to 0.$$

Thus, from (A20) we obtain (A18). Similarly, using $\psi_1'(0,-\infty)=0$ and $\psi_1(0,-\infty)=\gamma$, we get (A19).

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[1] M. Jaulent and C. Jean, ''The inverse s-wave scattering problem for a class of potentials depending on energy,'' Commun. Math. Phys. **28**, 177–220 (1972).
[2] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. I,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 105–118 (1976).
[3] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. II,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 119–137 (1976).
[4] M. Jaulent, ''Inverse scattering problems in absorbing media,'' J. Math. Phys. **17**, 1351–1360 (1976).
[5] D. H. Sattinger and J. Szmigielski, ''Energy dependent scattering theory,'' Differ. Integral Eqs. **8**, 945–959 (1995).
[6] D. J. Kaup, ''A higher-order water-wave equation and the method for solving it,'' Prog. Theor. Phys. **54**, 396–408 (1975).
[7] M. Tsutsumi, ''On the inverse scattering problem for the one-dimensional Schrödinger equation with an energy dependent potential,'' J. Math. Anal. Appl. **83**, 316–350 (1981).
[8] D. H. Sattinger and J. Szmigielski, ''A Riemann-Hilbert problem for an energy dependent Schrödinger operator,'' Inverse Probl. **12**, 1003–1025 (1996).
[9] L. D. Faddeev, ''Properties of the S-matrix of the one-dimensional Schrödinger equation,'' Am. Math. Soc. Trans. **2**, 139–166 (1964) [Trudy Mat. Inst. Steklova **73**, 314–336 (1964) (Russian)].
[10] P. Deift and E. Trubowitz, ''Inverse scattering on the line,'' Commun. Pure Appl. Math. **32**, 121–251 (1979).
[11] M. Klaus, ''Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line,'' Inverse Probl. **4**, 505–512 (1988).
[12] T. Aktosun, M. Klaus, and C. van der Mee, ''Inverse wave scattering with discontinuous wave speed,'' J. Math. Phys. **36**, 2880–2928 (1995).
[13] V. N. Pivovarchik, ''On the discrete spectrum of a boundary value problem,'' Sib. Math. J. **31**, 853–856 (1990) [Sibirsk. Mat. Zh. **31**, 182–186 (1990) (Russian)].
[14] V. N. Pivovarchik, ''Eigenvalues of a certain quadratic pencil of operators,'' Funct. Anal. Appl. **23**, 70–72 (1989) [Funkt. Anal. i ego Priloz. **23**, 80–81 (1989) (Russian)].
[15] V. N. Pivovarchik, ''On the total algebraic multiplicity of the spectrum in the right-half plane for a quadratic operator pencil,'' St. Petersburg Math. J. **3**, 447–454 (1992) [Algebra i Analiz **3**, 223–230 (1991) (Russian)].
[16] V. N. Pivovarchik, ''On positive spectra of one class of polynomial operator pencils,'' Integr. Eqs. Oper. Th. **19**, 314–326 (1994).
[17] R. P. Boas, Entire Functions (Academic, New York, 1954).
[18] D. B. Hinton, M. Klaus, and J. K. Shaw, ''Levinson's theorem and Titchmarsh-Weyl $m(\lambda)$ theory for Dirac systems,'' Proc. R. Soc. Edinburgh Sect. A **109**, 173–186 (1988).
[19] I. C. Gohberg and E. I. Sigal, ''An operator generalization of the logarithmic residue theorem and the theorem of Rouché,'' Math. USSR Sbornik **13**, 603–625 (1971) [Mat. Sbornik **84**, 607–629 (1971) (Russian)].
[20] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations (McGraw-Hill, New York, 1955).
[21] N. Levinson, ''On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase,'' Danske Vid. Selsk. Mat.-Fys. Medd. **25**, 1–29 (1949).
[22] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Mathematics, 1258 (Springer, Berlin, 1987).
[23] M. Reed and B. Simon, Methods of Modern Mathematical Physics. IV. Analysis of Operators (Academic, New York, 1978).
[24] M. Reed and B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness (Academic, New York, 1975).
[25] T. Kato, Perturbation Theory for Linear Operators, 2nd ed. (Springer, Berlin, 1976).
[26] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw-Hill, New York, 1978).