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BASIC PROPERTIES OF MATRICES DESCRIBING SCATTERING OF POLARIZED LIGHT IN ATMOSPHERES AND OCEANS.

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ABSTRACT

Mueller matrices and Cloude coherency matrices are used to describe the transfer of polarized light in atmospheres, oceans and atmosphere-ocean systems. Basic properties of such matrices are presented for single and multiple scattering as well as reflection and refraction by interfaces and boundary surfaces.

1. INTRODUCTION

Transfer of polarized light in atmospheres and oceans can conveniently be described by means of Stokes parameters I, Q, U and V, which represent the radiance (or flux) and state of polarization of a beam of (quasi-)monochromatic radiation [Van de Hulst 1957, 1980]. When these Stokes parameters are written as elements of a column vector each linear change of this vector may be characterized by means of a real 4 x 4 matrix that transforms the vector into a similar column vector of four Stokes parameters. Such a matrix is called a Mueller matrix. Examples are provided by matrices describing single or multiple scattering and matrices corresponding to the reflection and refraction of light at a surface of discontinuity, such as the atmosphere-ocean interface.

Quite often the 16 elements of a Mueller matrix obey certain interrelationships (equalities and inequalities). In recent years the literature on such basic properties has grown rapidly. The primary purpose of this paper is twofold, namely (i) to briefly summarize some of the most important basic properties of matrices describing single scattering of polarized light in atmospheres and oceans and (ii) to present some results for matrices describing multiple scattering in such media including boundary effects. Within the limited framework of this paper we will have to make ample use of references for details.

2. PURE MUELLER MATRICES

Consider a monochromatic plane wave that is scattered in an arbitrary direction by an arbitrary particle in a fixed orientation. This process can be described by

$$\begin{pmatrix} E_l^{sc} \\ E_r^{sc} \end{pmatrix} = A \begin{pmatrix} E_l^{in} \\ E_r^{in} \end{pmatrix}, \tag{1}$$

where A is called the (scattering) amplitude matrix. Here E_l^{sc} and E_r^{sc} denote the electric field components of the scattered wave, parallel and perpendicular, respectively, to the plane containing the directions of the incident and scattered beams. Similarly, E_l^{in} and E_r^{in} describe the incident beam. In terms of Stokes parameters we have for the same process

$$\mathbf{I}^{sc} = \mathbf{F}\ \mathbf{I}^{in}, \tag{2}$$

where the bold letters $\mathbf{I}$ are column vectors whose elements are the Stokes parameters, and the real 4 x 4 matrix $\mathbf{F}$ is the scattering matrix of a single particle. This matrix is a pure Mueller matrix, which means that its elements can be expressed in the elements of a 2 x 2 amplitude matrix, by applying the definition of Stokes parameters on both sides of Eq. (1). The explicit expressions have been reported by Van de Hulst [1957, section 5.14]. However, the relationship between $\mathbf{F}$ and A can also be expressed by a matrix relation involving a Kronecker product [see e.g. Barakat, 1981]. The matrix $\mathbf{F}$ can also be used for single scattering of quasi-monochromatic light. From hereon we will only consider this type of light, unless explicitly stated otherwise.

Other examples of pure Mueller matrices are the matrices describing the changes of Stokes parameters upon reflection and refraction by a smooth interface between two isotropic, nonconducting media. These matrices are readily obtained from the Fresnel formulae, since the underlying amplitude matrix is diagonal [see e.g. Born and Wolf, 1964, section 1.5; Tsang et al., 1985; Kattawar and Adams, 1989].

For each (nonvanishing) pure Mueller matrix 9 independent interrelations for its elements exist which can be used to derive all other relations based on the existence of one underlying amplitude matrix [see Van de Hulst, 1957, section 5.14; Hovenier et al., 1986]. For a comprehensive discussion of the structure of a general pure Mueller matrix we refer to Hovenier (1994) and references therein. If a real 4 x 4 matrix is obtained from experiments or calculations and one wishes to know if this can be a pure Mueller

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matrix a variety of tests can be used [see Hovenier and Van der Mee, 1996 and references therein].

For later applications it is useful to mention the following two general theorems for pure Mueller matrices [see Hovenier, 1994].
Theorem 1: If M^p is a pure Mueller matrix and c is a real non-negative scalar, then the product cM^p is a pure Mueller matrix.
Theorem 2: If M_1^p and M_2^p are both pure Mueller matrices, then their product $M_1^p M_2^p$ is a pure Mueller matrix.

Quite often it is useful to employ Dirac delta functions. Therefore, we state explicitly that the product of a delta function and a matrix shall be called a pure Mueller matrix if and only if the matrix occurring in the product is a pure Mueller matrix.

3. SUMS OF PURE MUELLER MATRICES

Let us consider the sum of pure Mueller matrices (SPM)

$$M = \sum_{k=1}^{N} M_k^p , \tag{3}$$

where k = 1, 2...and N is a positive integer $(N \geq 1)$. For $N \geq 2$ the structure of M is, generally, much less involved than that of a pure Mueller matrix. For every SPM we have the following six elementary inequalities [Fry and Kattawar, 1981; Hovenier et al., 1986]

$$\begin{aligned}(M_{11} \pm M_{22})^2 - (M_{12} \pm M_{21})^2 \geq \\ (M_{33} \pm M_{44})^2 + (M_{34} \mp M_{43})^2\end{aligned} \tag{4}$$

$$\begin{aligned}(M_{11} \pm M_{12})^2 - (M_{21} \pm M_{22})^2 \geq \\ (M_{31} \pm M_{32})^2 + (M_{41} \pm M_{42})^2\end{aligned} \tag{5}$$

$$\begin{aligned}(M_{11} \pm M_{21})^2 - (M_{12} \pm M_{22})^2 \geq \\ (M_{13} \pm M_{23})^2 + (M_{14} \pm M_{24})^2\end{aligned} \tag{6}$$

where M_{ij} is the element of the i-th row and j-th column of M and i,j = 1,2,3,4.

By taking linear combinations of the elements of M, Cloude (1986) constructed an alternative matrix, T, which he called the coherency matrix [see also Van der Mee, 1993; Hovenier and Van der Mee, 1996]. An important property of this coherency matrix is that it has only real, non-negative eigenvalues if $N \geq 2$ and only one non-vanishing positive eigenvalue if $N = 1$ (i.e. for a pure non-vanishing Mueller matrix). Since these conditions are necessary and sufficient they provide excellent means for testing purposes. Instead of T one can also use a matrix introduced by Simon (1982), which is unitarily equivalent to T.

It follows directly from theorems 1 and 2 of section 2 that multiplying an SPM by a non-negative scalar, or a pure Mueller matrix, or an SPM results in another SPM. We will call this the multiplication rule for SPM's. Further, it is clear from Eq. (3) and properties of integrals that the integral of a pure Mueller matrix over any domain of real variables has exactly the same properties as an SPM. Such a real variable may concern the particle size and/or orientation of particles and surface elements as well as direction and wavelength.

It is well-known that the scattering matrix of a collection of independently scattering particles is an SPM [Van de Hulst, 1957, section 5.21]. This is the common situation for single scattering by a small volume element in an atmosphere or ocean. The total radiation field in such a medium is, however, also determined by other processes, such as multiple scattering as well as reflection and refraction by boundary surfaces. This will be considered in the following sections.

4. PLANE-PARALLEL ATMOSPHERES

Consider a plane-parallel atmosphere bounded below by a solid or liquid surface that does not transmit light. The atmosphere is illuminated at the top by a parallel beam of polarized light. This beam is described by the vector $\pi\Phi$, which is normalized so that its first element is the flux per unit horizontal area of the atmosphere. The incoming light is scattered in the atmosphere by randomly oriented particles, each of which has a plane of symmetry, (or by particles and their mirror images in equal numbers and in random orientation). Thus the extinction can be fully described by means of a positive scalar [see Van de Hulst, 1957, section 5.41].

We use the local meridian plane as the plane of reference for the Stokes parameters [Chandrasekhar, 1950, section 17.1]. The radiation at optical depth τ (measured from the top downwards) may be written as

$$I(\tau, u, \varphi) = K(\tau, u, \varphi, \mu_0, \varphi_0)\Phi . \tag{7}$$

Here the first element of $I(\tau, u, \varphi)$ is the diffuse radiance in a direction given by the angle with the downward normal (whose cosine is u) and the azimuth angle φ. Similarly, μ_0 and φ_0 specify the direction of the beam incident at the top of the atmosphere. In downward directions we have, in addition to the diffuse radiation, the reduced incident radiation $\exp(-\tau/\mu_0)\Phi$, which represents light that has not

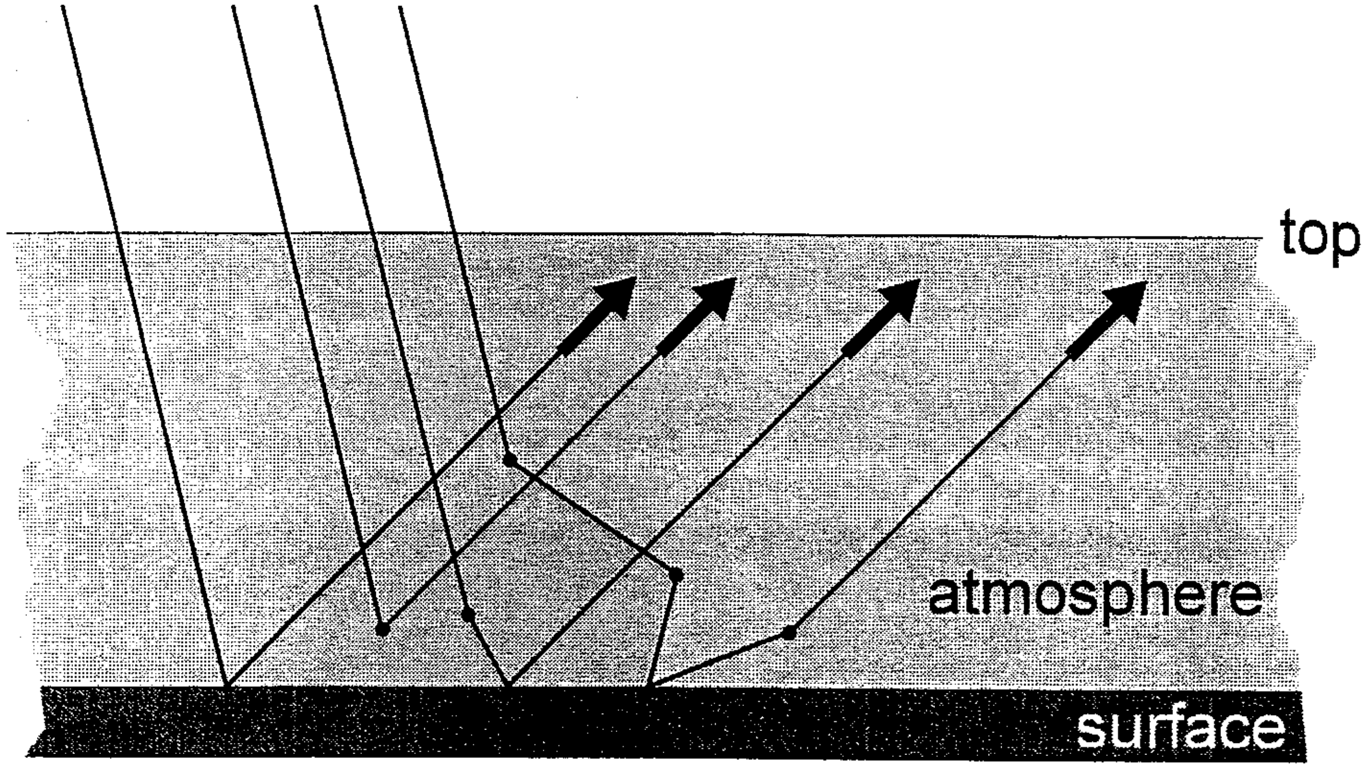

Figure 1.

Schematic representation of scattering in an atmosphere and reflection by an underlying surface. A parallel beam of light illuminates the top of the atmophere. Thin lines represent light paths. The thick arrows represent contributions to the Stokes parameters at an arbitrary optical depth in an arbitrary direction.

been scattered in the atmosphere nor reflected by the underlying surface. In the special case of a perfectly absorbing ground surface the real 4 x 4 matrix K is usually called the reflection matrix if $\tau = 0$, and the diffuse transmission matrix if τ has its maximal value. For intermediate values of the optical depth in this special case, K is equivalent to the matrices U and D introduced by Hovenier and De Haan (1985) for the internal upward and diffuse downward radiation, respectively. If the surface is not perfectly absorbing we assume that reflection by the surface alone can be described analogous to Eq. (7) by the surface reflection matrix R^s.

To investigate the structure of K we study the history of the light that is represented by $I(\tau, u, \varphi)$ [see also Fig. 1]. Clearly we can write

$$K = K^a + K^s + K^{as} \tag{8}$$

where:

- K^a represents light that has been scattered one or more times in the atmosphere without ever reaching the surface,
- K^s represents light reflected by the surface without ever being scattered in the atmosphere, and
- K^{as} represents light which is due to combinations of scattering in the atmosphere and reflection by the underlying surface.

First we consider the contribution to $I(\tau, u, \varphi)$ due to K^a. Naturally, extinction may occur before each scattering and after the last scattering. If the scattering by an arbitrary particle can be described by a matrix F, as in Eq. (2), we must pre- and postmultiply this matrix by rotation matrices, because the local meridian plane is now the plane of reference for the Stokes parameters. The result is the matrix

$$Z = L(\sigma_2) F L(\sigma_1), \tag{9}$$

where

$$L(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\sigma & \sin 2\sigma & 0 \\ 0 & -\sin 2\sigma & \cos 2\sigma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The latter matrix is a pure Mueller matrix, whose amplitude matrix is obtained from the middle block by writing σ instead of 2σ. Hence Z is a pure Mueller matrix. We can now conclude that we can write

$$K^a = \sum_{j=1}^{\infty} K_j^a, \qquad (11)$$

where the index j corresponds to the order of scattering and each K_j^a is a pure Mueller matrix, so that K^a is an SPM. Let us also assume that R^s is an SPM (including the case of one pure Mueller matrix). Then the same is true for K^s since positive scalars (due to extinction) and rotation matrices are immaterial in this respect. Finally, K^{as} is an SPM, as follows from the multiplication rule for SPM's, discussed in section 3. Consequently, Eq. (8) yields that K is an SPM. This concludes our investigation of the structure of the matrix $K(\tau, u, \varphi, \mu_0, \varphi_0)$ occurring in Eq. (7).

5. OCEANS AND OCEAN-ATMOSPHERE SYSTEMS

If we replace the atmosphere considered in the preceding section by a scattering ocean with scalar extinction, a completely analogous treatment can be given. The only difference is that we must take diffraction and internal reflection at the top boundary into account. Since these effects can be described by pure Mueller matrices (for a smooth interface) or by an SPM (for a rough interface considered to be made up of locally smooth facets) we can draw the same conclusion as before, namely that we can use Eq. (7) in which $K(\tau, u, \varphi, \mu_0, \varphi_0)$ is an SPM.

It is now readily verified that if we have an atmosphere above an ocean, and both have scalar extinction, we can again use Eq. (7) with $K(\tau, u, \varphi, \mu_0, \varphi_0)$ being an SPM. Of course, this remains true if the atmosphere - ocean system is bounded below by a reflecting surface which can be characterized itself by a matrix which is an SPM.

6. DISCUSSION AND CONCLUSIONS

The radiance and state of polarization of the radiation inside and outside a plane-parallel atmosphere, ocean or atmosphere-ocean system can be expressed in terms of a Mueller matrix. We have shown that, in general, this matrix is a sum of a pure Mueller matrices. Properties of these matrices and their corresponding Cloude coherency matrices have been discussed. These properties can be used for several purposes, in particular to test computer codes and experimental results [see also Hovenier and Van der Mee, 1996].

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REFERENCES

Barakat, R., 1981: Bilinear Constraints between Elements of the 4 x 4 Mueller-Jones Transfer Matrix of Polarization Theory. *Opt. Commun.* **38**, 159-161.

Born M. and E. Wolf, 1964: *Principles of Optics*, Pergamon Press, Oxford.

Chandrasekhar, S., 1950: *Radiative Transfer*, Oxford Univ. Press, London; also 1960: Dover, New York.

Cloude, S.R., 1986: Group Theory and Polarisation Algebra. *Optik* **75**, 26-36.

Fry, E.S. and G.W. Kattawar, 1981: Relationships between Elements of the Stokes Matrix. *Appl. Optics* **20**, 2811-2814.

Hovenier, J.W., 1994: Structure of a general pure Mueller matrix. *Appl. Optics* **33**, 8318 - 8324 .

Hovenier J.W. and J.F. de Haan, 1985: Polarized light in planetary atmospheres for perpendicular directions. *Astron. Astrophys.* **146**, 185-191.

Hovenier, J.W., H.C. van de Hulst and C.V.M. van der Mee 1986: Conditions for the Elements of the Scattering Matrix. *Astron. Astrophys.* **157**, 301-310.

Hovenier, J.W. and C.V.M. van der Mee, 1996: Testing scattering matrices: a compendium of recipes. *J. Quant. Spectrosc. Radiat. Transfer* **55**, 649-661.

Hulst, H.C. van de, 1957: *Light Scattering by Small Particles*. Wiley, New York; also: 1981: Dover, New York.

Hulst, H.C. van de, 1980: *Multiple Light Scattering*. Academic Press, New York.

Kattawar, G.W. and C.N. Adams,1989: Stokes vector calculations of the submarine light field in an atmosphere-ocean with scattering according to a Rayleigh phase matrix: Effect of interface refractive index on radiance and polarization. *Limnol. Oceanogr.* **34**, 1453-1472.

Mee, C.V.M. van der, 1993: An Eigenvalue Criterion for Matrices Transforming Stokes Parameters. *J. Math. Phys.* **34**, 5072-5088.

Simon, S., 1982: The Connection between Mueller and Jones Matrices of Polarization Optics. *Opt. Commun.* **42**, 293-297.

Tsang, L., J.A Kong and R.T. Shin, 1985: *Theory of Microwave Remote Sensing*. Wiley, New York.