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ANALYTIC OPERATOR FUNCTIONS WITH COMPACT SPECTRUM. I. SPECTRAL NODES, LINEARIZATION AND EQUIVALENCE

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This paper arose from an attempt to classify analytic operator functions modulo equivalence in terms of their linearizations and to use the linearization as a tool to obtain spectral factorizations. In this first part spectral linearizations and spectral nodes are introduced to provide a general framework to deal with problems concerning the uniqueness of a linearization and the existence of analytic divisors. Two analytic operator functions $W_1(.)$ and $W_2(.)$ with compact spectrum are shown to have similar spectral linearizations if and only if for some Banach space Z the functions $W_1(.) \oplus I_Z$ and $W_2(.) \oplus I_Z$ are equivalent. In parts II and III of this paper spectral nodes will be used intensively to deal with a number of factorization problems. In particular, in part III for Hilbert spaces and bounded domains a full solution of the inverse problem will be given, which will be used to construct spectral factorizations explicitly and to solve the problem of spectrum displacement.

0. Introduction

First let us recall some known notions and facts concerning equivalence and linearization. Let Ω be an open set in $\mathbb{C}$, and let $W_1 : \Omega \to L(Y_1)$ and $W_2 : \Omega \to L(Y_2)$ be analytic operator-valued

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functions. Here $L(Y_i)$, $i = 1,2$, stands for the Banach algebra of all (bounded linear) operators acting on the complex Banach space Y_i. The functions W_1 and W_2 are called *equivalent on* Ω if

$$W_1(\lambda) = E(\lambda)W_2(\lambda)F(\lambda) \ , \ \lambda \in \Omega \ ,$$

where $E_1(\lambda) : Y_2 \to Y_1$ and $F(\lambda) : Y_1 \to Y_2$ are (two-sided) invertible operators which depend analytically on $\lambda \in \Omega$. (In this case Y_1 and Y_2 are necessarily isomorphic.) A (bounded linear) operator $T\colon U \to U$, where U is a Banach space, is called a *linearization* of the analytic operator function $W\colon \Omega \to L(Y)$ if for some Banach space Z the operator functions $W(\lambda) \oplus I_Z$ and $\lambda I_U - T$ are equivalent on Ω. Here I_Z and I_U stand for the identity operators on the spaces Z and U, respectively. The operator function $W(.) \oplus I_Z$ will be referred to as the *Z-extension* of W.

A systematic study of the problem to classify analytic operator functions modulo equivalence was started in [9,10]. In these papers the linearization was introduced and proved to be an important tool in the study of operator functions. Later papers about linearization ([22,5,18]) concerned mainly the existence of linearization and various explicit formulas for it. In the present paper we come back to one of the main themes of [9,10]. Can linearization be used to classify operator functions up to equivalence and extension? In general the answer is no. A given analytic operator function may have many different non-similar linearizations. Nevertheless for operator functions with compact spectrum the problem has a positive solution provided spectral linearizations are used only.

Let $W\colon \Omega \to L(Y)$ be an analytic operator function, and assume that the set

$$(0.1) \qquad \Sigma(W) = \{\lambda \in \Omega \mid W(\lambda) \text{ is not invertible}\}$$

is a compact set of Ω. We call the set $\Sigma(W)$ the *spectrum* of W in Ω. An operator $A\colon X \to X$, where X is a Banach space, is said to be a *spectral linearization* of W on Ω if the spectrum $\sigma(A)$ of the operator A is a subset of Ω and for some Banach spaces Z_1 and Z_2 the operator functions $W(\lambda) \oplus I_{Z_1}$ and $(\lambda I_X - A) \oplus I_{Z_2}$ are equivalent on Ω. A spectral linearization of W always exists provided the spectrum $\Sigma(W)$ is compact, and if Y is a Hilbert space, then the domain of a spectral linearization can be chosen to be a Hilbert space too.

The linearizator, introduced and studied in [20], is an example of a spectral linearization. If $W(\lambda) = \Sigma_{j=0}^{\ell} \lambda^j A_j$ is a monic (i.e., its leading coefficient A_ℓ is equal to the identity operator I) operator polynomial, then its companion operator

$$
\begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & I \\ -A_0 & -A_1 & -A_2 & \dots & -A_{\ell-1} \end{bmatrix} \tag{0.2}
$$

is a spectral linearization of W on $\mathbb{C}$ (see [12, 13, 14] for more details). A Hilbert space contraction T with spectral radius less than one turns out to be a spectral linearization of the holomorphic function $U\Theta_T$ on the open unit disc, where Θ_T is the Sz-Nagy-Foias characteristic operator function of T (cf. [28]) and U is a suitable fixed unitary operator.

Spectral linearizations classify analytic operator functions with a compact spectrum up to equivalence and extension. This is the contents of the next theorem, which is one of the main results of the present paper.

THEOREM 0.1. *Let $W_1, W_2 : \Omega \to L(Y)$ be analytic operator functions with compact spectra. Then there exist Banach spaces Z_1 and Z_2 such that $W_1(\lambda) \oplus I_{Z_1}$ and $W_2(\lambda) \oplus I_{Z_2}$ are equivalent on Ω if and only if W_1 and W_2 have similar spectral linearizations.*

In particular one sees that for a given operator function the spectral linearization is uniquely determined up to similarity. In other words we have the following corollary.

COROLLARY 0.2. *Let X_1 and X_2 be Banach spaces, and let $T_i \in L(X_i)$, $i = 1,2$. Then for some Banach spaces Z_1 and Z_2 the functions $(\lambda I_{X_1} - T_1) \oplus I_{Z_1}$ and $(\lambda I_{X_2} - T_2) \oplus I_{Z_2}$ are equivalent on an open set containing the spectra of T_1 and T_2 if and only if T_1 and T_2 are similar.*

In case $Z_1 = Z_2 = 0$ Corollary 0.2 was proved in [25], but the method used in [25] is not strong enough to obtain Corollary

0.2. The similarity referred to in Corollary 0.2 will be described explicitly in terms of the operator functions that give the equivalence between $(\lambda I_{X_1} - T_1) \oplus I_{Z_1}$ and $(\lambda I_{X_2} - T_2) \oplus I_{Z_2}$.

Theorem 0.1 is proved in this paper as a corollary of a general theory of spectral nodes. Let Ω be an open set in $\mathbb{C}$, and let $W: \Omega \to L(Y)$ be an analytic operator function with compact spectrum in Ω. A quintet $\theta = (A,B,C;X,Y)$ is called a *spectral node* for W on Ω if X is a Banach space,

$$A: X \to X \ , \quad B: Y \to X \ , \quad C: X \to Y$$

are (bounded linear) operators and the following conditions are satisfied:

(P_1) $\sigma(A) \subset \Omega$;

(P_2) $W(\lambda)^{-1} - C(\lambda I - A)^{-1}B$ has an analytic extension on Ω;

(P_3) $W(\lambda)\, C(\lambda I - A)^{-1}$ has an analytic extension on Ω ;

(P_4) $\cap_{j=0}^{\infty} \operatorname{Ker} CA^j = (0)$.

The operator A will be referred to as the *main operator* of the spectral node θ.

The notion of a spectral node is a natural generalization of the notions of standard triples and Γ-spectral triples for operator polynomials, which have been introduced and studied in [12, 13, 14, 16, 27]. On the other hand, spectral nodes are related to realizations for analytic operator functions (cf. [3], Section 2.3; also [11], Section III.1).

Linearization theorems of [10] are used to establish the existence of spectral nodes. Spectral nodes for a given W are unique up to similarity, i.e., if $(A_1,B_1,C_1;X_1,Y)$ and $(A_2,B_2,C_2;X_2,Y)$ are spectral nodes for W on Ω, then

$$A_1 = S^{-1}A_2S \ , \quad C_1 = C_2S \ , \quad B_1 = S^{-1}B_2$$

for some invertible operator S. Furthermore, we show that the main operator in a spectral node is a spectral linearization of W on Ω, and, conversely, every spectral linearization of W on Ω is the main operator in some spectral node for W. Note that these properties already prove Corollary 0.2.

In part II of this paper the connection between the invariant subspaces of the spectral linearization and analytic divisors will be explained. As applications the stability of spectral factorization will be proved and necessary and sufficient conditions will be obtained for the existence of Wiener-Hopf factorizations in terms of the moments of W^{-1} (cf. [27], also [11], Section III.1). In part III for Hilbert spaces and bounded domains the main inverse problem will be solved completely, and this will be used to construct spectral factorizations explicitly without using the cocycle theory (see, for instance, [15]). Furthermore, in part III the problem of spectrum displacement will be solved and we give applications to the theory of characteristic operator functions.

Let us now describe the contents of the various sections of this first part. In Section 1 we prove the uniqueness of spectral nodes. In Section 2 a calculus of spectral nodes is developed which is comparable to the operational calculus for operators. Explicit formulas are given for the spectral nodes of a product and a direct sum of two operator functions. The connection between the spectral nodes for an operator function W and the spectral nodes for an extension $W(\cdot) \oplus I_Z$ is described. Also, in this section, we describe the effect on the spectral nodes of a number of standard operations on operator functions, such as taking duals, applying Möbius transformations (cf. [3], Section 1.5) and passing to the hull of an operator function. In Section 3 the existence of spectral nodes is established and the connection with linearization is described. In Section 4 spectral nodes are characterized in terms of invertibility of operator matrices of the form

$$\begin{bmatrix} A - \lambda & B \\ C & * \end{bmatrix}, \qquad \lambda \in \Omega .$$

As a corollary we obtain that the main operator of a spectral node is a spectral linearization. Further, we give necessary and sufficient conditions in order that the quintet $(A,B,C;X,Y)$ is a spectral node for an analytic function $W\colon \Omega \to L(Y)$ such that $W(\lambda)^{-1}$ has an analytic continuation outside $\Sigma(W)$ including the point ∞, and we show how in that case the function may be reconstructed from the spectral node. In Section 5 we prove Theorem 0.1 mentioned above. Further, in this section, for separable Hilbert space we give a complete description of the minimal extensions that are needed to make equivalent two operator functions that have similar spectral linearizations.

We conclude with a few remarks about notation and terminology. We use the symbol $\mathbb{C}_\infty$ to denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. By a bounded Cauchy domain Δ we mean a bounded open set in $\mathbb{C}$ whose boundary consists of a finite number of disjoint, closed, rectifiable Jordan curves that are oriented in the positive sense. If Ω is an open subset of $\mathbb{C}$ and σ a compact set in Ω, then one can always find a bounded Cauchy domain Δ such that $\sigma \subset \Delta \subset \overline{\Delta} \subset \Omega$ (see [24], Section 148). The symbols X, Y, Z denote complex Banach spaces. The Banach space of all (bounded linear) operators between X and Y is denoted by $L(X,Y)$; if $X = Y$, we write $L(X)$. Throughout this paper Ω is an open subset of $\mathbb{C}$, and

$$W: \Omega \to L(Y)$$

is an analytic operator function whose spectrum $\Sigma(W)$, which is defined by (0.1), is a compact subset in Ω. All operators are assumed to be bounded and linear.

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1. Uniqueness of Spectral Nodes

In this section we show that for a given analytic operator function spectral nodes, whenever they exist, are unique up to similarity. First we derive the dual of Condition (P_3) for a spectral node.

PROPOSITION 1.1. *If* $\theta = (A,B,C;X,Y)$ *is a spectral node for* W *on* Ω*, then* $(\lambda - A)^{-1}BW(\lambda)$ *has an analytic continuation to* Ω.

Proof. Let Δ be a bounded Cauchy domain such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. We have to show that for each $y \in Y$

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{(\lambda-A)^{-1}BW(\lambda)}{z-\lambda}\, y\,d\lambda = 0 \qquad (z \in \Omega \setminus \overline{\Delta}).$$

Put $H(\lambda) = W(\lambda)^{-1} - C(\lambda - A)^{-1}B$, $\lambda \in \Omega \setminus \Sigma(W)$. Note that H has an analytic continuation to Ω (cf. Condition (P_2)), while

$$C(\lambda - A)^{-1}BW(\lambda) = I - H(\lambda)W(\lambda). \tag{1.1}$$

By induction one proves that

$$A^n\varphi(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{\lambda^n(\lambda-A)^{-1}BW(\lambda)}{z-\lambda}\, y\,d\lambda \quad (z \in \Omega \setminus \overline{\Delta},\ n \geq 0).$$

It follows that for each $z \in \Omega \setminus \overline{\Delta}$

$$CA^n\varphi(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{\lambda^n[I-H(\lambda)W(\lambda)]}{z-\lambda}\, y\,d\lambda = 0 \quad (n \geq 0) .$$

Using (P_4) one sees that $\varphi(z) = 0$, $z \in \Omega \setminus \overline{\Delta}$. □

THEOREM 1.2. *For* $i = 1,2$, *let* $\theta_i = (A_i, B_i, C_i; X_i, Y)$ *be a spectral node for* W *on* Ω. *Then there exists a unique invertible operator* $S\colon X_1 \to X_2$ *with the property that*

$$C_2S = C_1, \quad A_2S = SA_1, \quad B_2 = SB_1 . \tag{1.2}$$

The similarity S *and its inverse* S^{-1} *are given by the formulas*

$$S = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda-A_2)^{-1}B_2W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda \ ; \tag{1.3a}$$

$$S^{-1} = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda-A_1)^{-1}B_1W(\lambda)C_2(\lambda-A_2)^{-1}d\lambda \ , \tag{1.3b}$$

where Δ *is a bounded Cauchy domain such that* $(\sigma(A_1) \cup \sigma(A_2)) \subset \Delta \subset \overline{\Delta} \subset \Omega$.

Proof. Let T be the operator defined by the right-hand side of (1.3b). Note that the definition of T and S does not depend on the particular choice of the Cauchy domain Δ. Therefore we choose a bounded Cauchy domain Δ' such that $\Delta \subset \overline{\Delta} \subset \Delta' \subset \overline{\Delta'} \subset \Omega$. Then

$$\begin{aligned} TS &= (2\pi i)^{-1}\int_{\partial\Delta'} (\mu-A_1)^{-1}B_1W(\mu)C_2(\mu-A_2)^{-1}S\,d\mu \\ &= (2\pi i)^{-2}\int_{\partial\Delta'}\left(\int_{\partial\Delta} (\mu-A_1)^{-1}B_1W(\mu)C_2(\mu-A_2)^{-1}(\lambda-A_2)^{-1}\cdot \right.\\ &\qquad \left. \cdot\, B_2W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda\right)d\mu \ . \end{aligned}$$

We use the resolvent identity to rewrite the integrand as

$$(\mu-A_1)^{-1}B_1W(\mu)C_2(\mu-A_2)^{-1}\,\frac{B_2W(\lambda)C_1(\lambda-A_1)^{-1}}{\lambda-\mu} -$$

$$\frac{(\mu-A_1)^{-1}B_1W(\mu)C_2}{\lambda-\mu}(\lambda-A_2)^{-1}B_2W(\lambda)C_1(\lambda-A_1)^{-1}.$$

Observe that for a fixed $\mu \in \partial\Delta'$ the first term is analytic in λ on Δ. It follows that the double integral of the first term is zero. To integrate the second term we interchange the order of integration. By Proposition 1.1 the function $(\mu-A_1)^{-1}B_1W(\mu)$ is analytic on Ω. It follows that

$$TS = (2\pi i)^{-1}\int_{\partial\Delta}(\lambda-A_1)^{-1}B_1W(\lambda)C_2(\lambda-A_2)^{-1}B_2W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda.$$

Now we use formula (1.1) together with Condition (P_3) and Proposition 1.1, and get

$$TS = (2\pi i)^{-1}\int_{\partial\Delta}(\lambda-A_1)^{-1}B_1W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda.$$

For $n \geq 0$ we multiply by the operator $C_1A_1^n$ from the left, apply the formula $A_1(\lambda-A_1)^{-1} = \lambda(\lambda-A_1)^{-1} - I$ and make use of Condition (P_3) n times. For $n \geq 0$ this yields

$$C_1A_1^nTS = (2\pi i)^{-1}\int_{\partial\Delta}\lambda^nC_1(\lambda-A_1)^{-1}B_1W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda.$$

Finally we apply formula (1.1) and Condition (P_3) once again and get

$$C_1A_1^nTS = C_1\cdot(2\pi i)^{-1}\int_{\partial\Delta}\lambda^n(\lambda-A_1)^{-1}d\lambda\ , \qquad n \geq 0\ .$$

As $\partial\Delta$ encloses $\sigma(A_1)$, we eventually get

$$C_1A_1^nTS = C_1A_1^n\ , \qquad n \geq 0\ .$$

Now we use Condition (P_4) and conclude that $TS = I_{X_1}$. In the same way it is shown that $TS = I_{X_2}$. Hence, S is invertible and $S^{-1} = T$.

• To prove (1.2) we use arguments exposed previously, namely formula (1.1), Proposition 1.1 and Condition (P_3). For instance, denoting $H_2(\lambda) = W(\lambda)^{-1} - C_2(\lambda-A_2)^{-1}B_1$ we have

$$C_2S = (2\pi i)^{-1}\int_{\partial\Delta}C_2(\lambda-A_2)^{-1}B_2W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda =$$

$$= (2\pi i)^{-1}\int_{\partial\Delta}C_1(\lambda-A_1)^{-1}d\lambda - (2\pi i)^{-1}\int_{\partial\Delta}H_2(\lambda)W(\lambda)C_1(\lambda-A_1)^{-1}d\lambda.$$

Because of (P_3) the second integral vanishes, and the first one equals C_1.

Finally, the uniqueness of the similarity is immediate from Property (P_4) and Eqs. (1.2). □

COROLLARY 1.3. *If* $\theta = (A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *and* Δ *is a bounded Cauchy domain such that* $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$, *then*

$$(2\pi i)^{-1} \int_{\partial\Delta} (\lambda - A)^{-1} BW(\lambda) C(\lambda - A)^{-1} d\lambda = I_X .$$

Proof. Apply the previous theorem for $\theta_1 = \theta_2 = \theta$, and remark that the similarity obtained is unique. □

Two spectral nodes $\theta_1 = (A_1,B_1,C_1;X_1,Y)$ and $\theta_2 = (A_2,B_2,C_2;X_2,Y)$ are said to be *similar* if there exists an invertible $S\colon X_1 \to X_2$ such that formula (1.2) holds true.

THEOREM 1.4. *Two analytic operator functions* W_1, W_2: $\Omega \to L(Y)$ *with compact spectrum have similar spectral nodes on* Ω *if and only if the function* $W_1(\lambda)^{-1} - W_2(\lambda)^{-1}$ *has an analytic continuation to* Ω.

Proof. Suppose that for $i = 1,2$ the quintet $\theta_i = (A_i,B_i,C_i;X_i,Y)$ is a spectral node for W_i on Ω, and assume that θ_1 and θ_2 are similar. Then there exists an invertible $S\colon X_1 \to X_2$ such that $C_2S = C_1$, $A_2S = SA_1$ and $B_2 = SB_1$. Then the function $W_1(\lambda)^{-1} - W_2(\lambda)^{-1} = [W_1(\lambda)^{-1} - C_1(\lambda - A_1)^{-1}B_1] - [W_2(\lambda)^{-1} - C_2(\lambda - A_2)^{-1}B_2]$ has an analytic continuation to Ω.

Conversely, assume that $W_1(\lambda)^{-1} - W_2(\lambda)^{-1}$ has an analytic continuation to Ω, and let $\theta = (A,B,C;X,Y)$ be a spectral node for W_1 on Ω. Obviously, $W_2(\lambda)^{-1} - C(\lambda - A)^{-1}B = [W_1(\lambda)^{-1} - C(\lambda - A)^{-1}B] + [W_2(\lambda)^{-1} - W_1(\lambda)^{-1}]$ has an analytic continuation to Ω. Further, the function $W_2(\lambda)C(\lambda - A)^{-1} = W_1(\lambda)C(\lambda - A)^{-1} - W_2(\lambda)[W_2(\lambda)^{-1} - W_1(\lambda)^{-1}]W_1(\lambda)C(\lambda - A)^{-1}$ has an analytic continuation to Ω. So Properties (P_1) - (P_4) hold true for the node and the operator function W_2. Hence, θ is also spectral node for W_2 on Ω. □

2. Calculus of Spectral Nodes

In this section a calculus of spectral nodes is developed. In the next section these results will be used to construct explicitly a spectral node for a given analytic operator function, starting from linear functions. First we derive a lemma that will play an essential role in what follows.

LEMMA 2.1. *Assume that for the quintet $(A,B,C;X,Y)$ the properties (P_1), (P_2) and (P_3) hold. Let x be an element of X such that $C(\lambda-A)^{-1}x$ vanishes on a neighbourhood of infinity. Then $C(\lambda-A)^{-1}x = 0$ for all $\lambda \notin \sigma(A)$.*

Proof. Write $h(\lambda) = C(\lambda-A)^{-1}x$, $\lambda \notin \sigma(A)$. Let Δ be a bounded Cauchy domain such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. It suffices to show that $h(\lambda) = 0$ for $\lambda \notin \Delta$. Let U be the unbounded connected component of $\mathbb{C}_\infty \setminus \overline{\Delta}$, and let $\Delta_1,\ldots,\Delta_m$ be the connected components of Δ. The closure of U has points in common with $\overline{\Delta}$. Assume $\lambda_0 \in \overline{U} \cap \overline{\Delta}_1$. As $h(\lambda)$ vanishes on U and h is analytic outside $\sigma(A)$, the function h vanishes on a neighbourhood of λ_0. But then the same is true for Wh. According to Property (P_3) the function Wh has an analytic continuation to Ω. Because Δ_1 is connected, Wh is zero on $\overline{\Delta}_1$. From Property (P_2) it follows that $\Sigma(W) \subset \sigma(A)$. So we may conclude that h vanishes on $\partial\Delta_1$. But this implies that h is zero on the unbounded component of $\mathbb{C}_\infty \setminus \bigcup_{j=2}^{n} \overline{\Delta}_j$, and we can repeat the argument. Proceeding in this way we obtain $h = 0$, and the proof is complete. □

In the same way we can show that, if $C(\lambda-A)^{-1}x$ vanishes on some non-empty open set in the complement of $\sigma(A)$, then the vector $C(\lambda-A)^{-1}x = 0$ for all $\lambda \notin \sigma(A)$.

THEOREM 2.2. *For $i = 1,2$, let $W_i: \Omega \to L(Y)$ be an analytic operator function with compact spectrum, and let $\theta_i = (A_i,B_i,C_i;X_i,Y)$ be a spectral node for W_i on Ω. Put*

$$(2.1)\qquad A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} R \\ B_2 \end{bmatrix}, \quad C = [\, C_1 \quad Q \,],$$

where the operators R and Q are defined by

$$R = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda-A_1)^{-1}B_1\{W_2(\lambda)^{-1} - C_2(\lambda-A_2)^{-1}B_2\}d\lambda\ ;$$

$$Q = (2\pi i)^{-1}\int_{\partial\Delta} \{W_1(\lambda)^{-1} - C_1(\lambda-A_1)^{-1}B_1\}C_2(\lambda-A_2)^{-1}d\lambda\ .$$

Here Δ is a bounded Cauchy domain such that $(\sigma(A_1) \cup \sigma(A_2)) \subset \Delta \subset \overline{\Delta} \subset \Omega$. Then $(A,B,C;X_1 \oplus X_2,Y)$ is a spectral node for $W = W_2W_1$ on Ω.

Proof. Since $\sigma(A) \subset (\sigma(A_1) \cup \sigma(A_2))$ it is clear that Property (P_1) holds. For $\lambda \notin (\sigma(A_1) \cup \sigma(A_2))$ we have

$$(\lambda-A)^{-1} = \begin{bmatrix} (\lambda-A_1)^{-1} & (\lambda-A_1)^{-1}B_1C_2(\lambda-A_2)^{-1} \\ 0 & (\lambda-A_2)^{-1} \end{bmatrix} .$$

It follows that for $\lambda \notin (\sigma(A_1) \in \sigma(A_2))$

$$C(\lambda-A)^{-1}B = C_1(\lambda-A_1)^{-1}R + C_1(\lambda-A_1)^{-1}B_1C_2(\lambda-A_2)^{-1}B_2 + Q(\lambda-A_2)^{-1}B_2.$$

Take $w \in \Omega \setminus \overline{\Delta}$. Using the resolvent identity and Property (P_2) for W_1 and W_2 we compute that

$$C_1(w-A_1)^{-1}R = (2\pi i)^{-1}\int_{\partial\Delta} (w-\lambda)^{-1}W_1(\lambda)^{-1}H_2(\lambda)d\lambda \ ;$$

$$Q(w-A_2)^{-1}B_2 = (2\pi i)^{-1}\int_{\partial\Delta} (w-\lambda)^{-1}H_1(\lambda)W_2(\lambda)^{-1}d\lambda \ ,$$

where for $i = 1,2$, $H_i(\lambda) = W_i(\lambda)^{-1} - C_i(\lambda-A_i)^{-1}B_i$. It follows that the functions

$$W_1(\lambda)^{-1}H_2(\lambda) - C_1(\lambda-A_1)^{-1}R, \quad H_1(\lambda)W_2(\lambda)^{-1} - Q(\lambda-A_2)^{-1}B$$

have an analytic continuation to Ω. Now

$$W(\lambda)^{-1} - C(\lambda-A)^{-1}B = W_1(\lambda)^{-1}W_2(\lambda)^{-1} - C(\lambda-A)^{-1}B$$

$$= \{W_1(\lambda)^{-1}H_2(\lambda) - C_1(\lambda-A_1)^{-1}R\} + \{H_1(\lambda)W_2(\lambda)^{-1} - Q(\lambda-A_2)^{-1}B_2\} - H_1(\lambda)H_2(\lambda), \quad \lambda \in \Omega \setminus (\sigma(A_1) \cup \sigma(A_2)) .$$

This shows that $W(\lambda)^{-1} - C(\lambda-A)^{-1}B$ has an analytic continuation to Ω too.

To derive (P_3), we first note that for $w \in \Omega \setminus \overline{\Delta}$

$$Q(w-A_2)^{-1} = (2\pi i)^{-1} \int_{\partial\Delta} (w-\lambda)^{-1}H_1(\lambda)C_2(\lambda-A_2)^{-1}d\lambda \ .$$

This implies that $H_1(\lambda)C_2(\lambda-A_2)^{-1} - Q(\lambda-A_2)^{-1}$ has an analytic continuation to Ω. Now

$$W(\lambda)C(\lambda-A)^{-1} = [V_1(\lambda),\ V_2(\lambda)] \ ,$$

where

$$V_1(\lambda) = W_2(\lambda)W_1(\lambda)C_1(\lambda-A_1)^{-1} \ ;$$

$$V_2(\lambda) = W_2(\lambda)C_2(\lambda-A_2)^{-1} + W(\lambda)\{Q(\lambda-A_2)^{-1} - H_1(\lambda)C_2(\lambda-A_2)^{-1}\}.$$

This implies that $W(\lambda)C(\lambda-A)^{-1}$ has an analytic continuation to Ω, and hence Property (P_3) holds true.

Finally, to prove (P_4), take $x=(x_1,x_2)\in\bigcap_{j=0}^{\infty}\operatorname{Ker} CA^j$. Then $C(\lambda-A)^{-1}x$ vanishes on a neighbourhood of infinity. Using Lemma 2.1 we conclude that $C(\lambda-A)^{-1}x$ is zero for $\lambda\notin\sigma(A)$. Take $\lambda\in\Omega\setminus(\sigma(A_1)\cup\sigma(A_2))$. As $C(\lambda-A)^{-1}x=0$, we have

$$C_1(\lambda-A_1)^{-1}x_1+C_1(\lambda-A_1)^{-1}B_1C_2(\lambda-A_2)^{-1}x_2+Q(\lambda-A_2)^{-1}x_2=0.$$

Multiplying from the left by $W_1(\lambda)$ and rearranging terms we get

$$C_2(\lambda-A_2)^{-1}x_2=-W_1(\lambda)C_1(\lambda-A_1)^{-1}x_1+ W_1(\lambda)[H_1(\lambda)C_2(\lambda-A_2)^{-1}-Q(\lambda-A_2)^{-1}]x_2 .$$

In this identity the left-hand side is analytic outside $\sigma(A_2)$, whereas the right-hand side has an analytic continuation to Ω. By Liouville's theorem we have $C_2(\lambda-A_2)^{-1}x_2=0$ outside $\sigma(A_2)$, and therefore $x_2=0$. But then $C_1(\lambda-A_1)^{-1}x_1=0$ on a neighbourhood of infinity, and therefore $x_1=0$. Hence $x=0$, and Property (P_4) has been established. □

If for $i=1,2$, the function W_i^{-1} has an analytic continuation to $\mathbb{C}_\infty\setminus\Sigma(W_i)$, then it is clear from Property (P_2) and Liouville's theorem that for $i=1,2$

$$W_i(\lambda)^{-1}=D_i+C_i(\lambda-A_i)^{-1}B_i$$

for some bounded operator $D_i\colon Y\to Y$. Applying the previous theorem one sees that in this case the node

$$\theta=\left(\begin{bmatrix}A_1 & B_1C_2\\ 0 & A_2\end{bmatrix},\begin{bmatrix}B_1D_2\\ B_2\end{bmatrix},\ [\,C_1 \quad D_1C_2\,];\ X_1\oplus X_2, Y\right)$$

is spectral node for $W=W_2W_1$ on Ω. In the terminology of [3], Chapter 1, the node θ is the product $\theta=\theta_2\theta_1$ of the nodes θ_2 and θ_1.

A result similar to Theorem 2.2 has been provided in [12] in the framework of operator polynomials (see also [2, 26]).

THEOREM 2.3. *Let $\theta=(A,B,C;X,Y)$ be a spectral node for W on Ω, and let $E(\lambda)\in L(Y)$ and $F(\lambda)\in L(Y)$ be invertible operators that depend analytically on $\lambda\in\Omega$. Put*

$$R_E = (2\pi i)^{-1}\int_{\partial\Delta}(\lambda - A)^{-1}BE(\lambda)^{-1}d\lambda\ ,$$

$$Q_F = (2\pi i)^{-1}\int_{\partial\Delta}F(\lambda)^{-1}C(\lambda - A)^{-1}d\lambda\ ,$$

where Δ *is a bounded Cauchy domain such that* $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. *Then* $(A, R_E, Q_F; X, Y)$ *is a spectral node for* EWF *on* Ω.

Proof. Remark that $(0,0,0;\{0\},Y)$ is a spectral node for both E and F on Ω. Now apply Theorem 2.2 twice and identify the new space $\{0\} \oplus X \oplus \{0\}$ with the original space X. □

THEOREM 2.4. *For* $i = 1,2$ *let* Y_i *be a complex Banach space,* $W_i: \Omega \to L(Y_i)$ *an analytic operator function with compact spectrum and* $\theta_i = (A_i, B_i, C_i; X_i, Y_i)$ *a spectral node for* W_i *on* Ω. *Then*

$$(A_1 \oplus A_2, B_1 \oplus B_2, C_1 \oplus C_2; X_1 \oplus X_2, Y_1 \oplus Y_2)$$

is a spectral node for $W_1 \oplus W_2$ *on* Ω.

THEOREM 2.5. *Let* Y *and* Z *be complex Banach spaces, and denote by* π *and* τ *the projection of* $Y \oplus Z$ *onto* Y *along* Z *and the natural embedding of* Y *into* $Y \oplus Z$, *respectively.*

If $(A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *then* $(A, B\pi, \tau C; X, Y \oplus Z)$ *is a spectral node for* $W(\cdot) \oplus I_Z$ *on* Ω. *If* $(\tilde{A}, \tilde{B}, \tilde{C}; X, Y \oplus Z)$ *is a spectral node for* $W(\cdot) \oplus I_Z$ *on* Ω, *then* $(\tilde{A}, \tilde{B}\tau, \pi\tilde{C}; X, Y)$ *is a spectral node for* W *on* Ω.

Proof. The first part is clear from the previous theorem and the fact that $(0,0,0;\{0\},Z)$ is a spectral node for $W_2(\lambda) = I_Z$ on Ω. Let us prove the second part. Obviously, Property (P_1) holds. Since $W(\lambda)\cdot\pi\tilde{C}(\lambda-\tilde{A})^{-1} = \pi\cdot(W(\lambda) \oplus I_Z)\tilde{C}(\lambda-\tilde{A})^{-1}$, and $W(\lambda)^{-1} - \pi\tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B}\tau = \pi[(W(\lambda)^{-1} \oplus I_Z) - \tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B}]\tau$ for $\lambda \in \Omega \setminus \sigma(\tilde{A})$, it is clear that Properties (P_2) and (P_3) hold too.

To establish the final Property (P_4), we assume that for some $x \in X$ the vector function $\pi\tilde{C}(\lambda-\tilde{A})^{-1}x = 0$ on a neighbourhood of infinity. As (P_1), (P_2) and (P_3) have been established already, we may apply Lemma 2.1 and infer that $\pi\tilde{C}(\lambda-\tilde{A})^{-1}x = 0$ for $\lambda \notin \sigma(\tilde{A})$. Let us denote by ρ the projection of $Y \oplus Z$ onto Z along Y. Put

$$\tilde{H}(\lambda) = [W(\lambda)^{-1} \oplus I_Z] - \tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B}, \quad \lambda \in \Omega \setminus \Sigma(W).$$

Applying ρ to this identity we see that $\rho\tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B}$ has an analytic continuation to Ω. But then $\rho\tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B} = 0$, $\lambda \notin \sigma(A)$. With the help of Corollary 1.3 we obtain

$$\rho\tilde{C} = (2\pi i)^{-1} \int_{\partial\Delta} \rho\tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B}[W(\lambda) \oplus I_Z]\tilde{C}(\lambda-\tilde{A})^{-1}d\lambda = 0,$$

where Δ is a bounded Cauchy domain such that $\sigma(\tilde{A}) \subset \Delta \subset \overline{\Delta} \subset \Omega$. So on a neighbourhood of infinity we have $\tilde{C}(\lambda-A)^{-1}x = \tau\pi\tilde{C}(\lambda-A)^{-1}x = 0$, and $x = 0$ because (P_4) holds for $(\tilde{A},\tilde{B},\tilde{C};X,Y\oplus Z)$. □

In the next proposition φ will denote the Mobius transformation

$$\varphi(\lambda) = (p\lambda + q)(r\lambda + s)^{-1},$$

where p, q, r and s are complex numbers and $ps - qr \neq 0$. We consider φ as a map from the Riemann sphere $\mathbb{C}_\infty$ into itself. The inverse map is given by

$$\varphi^{-1}(\lambda) = (-s\lambda + q)(r\lambda - p)^{-1}.$$

In analogy with Theorem 1.9 of [3] we have

THEOREM 2.6. *Put* $\tilde{\Omega} = \varphi^{-1}[\Omega] \setminus \{\infty\}$. *Let* $\theta = (A,B,C;X,Y)$ *be a spectral node for* W *on* Ω, *and suppose* $T = p - rA$ *is invertible. Put*

$$\tilde{A} = -(q-sA)T^{-1}, \; \tilde{B} = T^{-1}B, \; \tilde{C} = (ps-rq)CT^{-1}.$$

Then $\tilde{\theta} = (\tilde{A},\tilde{B},\tilde{C};\tilde{X},\tilde{Y})$ *is a spectral node on* $\tilde{\Omega}$ *for the operator function*

$$\tilde{W}(\lambda) = W(\varphi(\lambda)), \quad \lambda \in \tilde{\Omega}.$$

Proof. As $T = p - rA$ is invertible, the inverse map φ^{-1} is analytic on the spectrum of A. So $\varphi^{-1}(A)$ is well-defined. In fact, $\varphi^{-1}(A) = -(q-sA)T^{-1}$ coincides with the main operator of $\tilde{\theta}$. By the spectral mapping theorem

$$\sigma(\varphi^{-1}(A)) = \{\lambda \in \mathbb{C}: \varphi(\lambda) \in \sigma(A)\} \subset \{\lambda \in \mathbb{C}: \varphi(\lambda) \in \Omega\} = \tilde{\Omega},$$

which establishes (P_1).

A straightforward calculation shows that for all $\lambda \in \tilde{\Omega} \setminus \sigma(\tilde{A})$

$$rCT^{-1}B + \tilde{C}(\lambda-\tilde{A})^{-1}\tilde{B} = C(\varphi(\lambda) - A)^{-1}B.$$

Since $\tilde{W}(\lambda) = W(\varphi(\lambda))$, $\lambda \in \tilde{\Omega}$, we now directly obtain (P_2).

Property (P_3) is clear from the identity

$$\tilde{C}(\lambda-\tilde{A})^{-1} = (r\lambda+s)^{-1}(ps-qr)C(\varphi(\lambda)-A)^{-1}, \lambda \in \tilde{\Omega} \quad ,$$

and the fact that $r\lambda + s \neq 0$ for all $\lambda \in \tilde{\Omega}$. From the same identity and Lemma 2.1 it follows that, if $\tilde{C}(\lambda-\tilde{A})^{-1}x = 0$ in a neighbourhood of infinity, then

$$C(\varphi(\lambda) - A)^{-1}x = 0 \quad , \qquad \lambda \in \tilde{\Omega} \setminus \sigma(\tilde{A}) \ .$$

Hence, $C(\zeta-A)^{-1}x = 0$ for all $\zeta \in \Omega \setminus \sigma(A)$. Using Property (P_4) for the spectral node θ, we conclude that $x = 0$. This establishes (P_4) for the quintet $\tilde{\theta}$, and the proof is complete. □

Next we consider a well-known construction for Banach spaces, which originates from [4]. Let X be a complex Banach space. We associate with X the Banach space $<X> = \ell_\infty(X)/c_0(X)$, which consists of all classes of bounded sequences $(x_n)_{n=0}^\infty$, $x_n \in X$ (two bounded sequences $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ are in the same class if and only if $\lim_{n\to\infty} \|x_n-y_n\| = 0$), and is endowed with the quotient norm

$$\|<(x_n)_{n=0}^\infty>\| = \operatorname{Inf}\left\{\operatorname{Sup}_{n\geq 0} \|x_n-z_n\| : \lim_{k\to\infty} z_k = 0\right\}.$$

Then one easily sees that $x \longrightarrow <(x,x,x,\ldots)>$ is an isometric embedding of X into $<X>$.

If X_1 and X_2 are complex Banach spaces, and $T: X_1 \to X_2$ a bounded linear operator, then the so-called *hull* of the operator T will be the operator $<T>: <X_1> \to <X_2>$, given by

$$<T> <(x_n)_{n=0}^\infty> = <(Tx_n)_{n=0}^\infty> \ .$$

One easily checks that this operator is well-defined and bounded. Further, the map $T \to <T>$ is a continuous linear transformation from $L(X_1,X_2)$ into the space $L(<X_1>,<X_2>)$. If $T \in L(X_1,X_2)$ and $S \in L(X_2,X_3)$, then

$$<ST> = <S> <T> \quad .$$

If X is a Banach space and $T \in L(X)$, we easily see that $\sigma(<T>) \subset \sigma(T)$ (in fact, $\sigma(<T>) = \sigma(T)$; see [19]). A Hilbert space analogue of the above construction can be found in [19].

We now state the following

THEOREM 2.7. *Let* $\theta = (A,B,C;X,Y)$ *be a spectral node for* W *on* Ω. *Then*

$$<\theta> = (<A>,<B>,<C> <X>,<Y>)$$

is a spectral node on Ω *for the operator function* $<W>$, *defined by*

$$<W>(\lambda) = <W(\lambda)> \ , \quad \lambda \in \Omega \ . \tag{2.2}$$

Proof. Because the map $T \to <T>$ is a continuous algebra homomorphism from the Banach algebra $L(Y)$ in $L(<Y>)$, it follows that the operator function $<W>$, defined by (2.2), is, indeed, analytic on Ω. Since $<I_Y> = I_{<Y>}$, it follows that $<W>$ has a compact spectrum which is contained in the spectrum of the operator function W. To see this, note that for $\lambda \in \Omega \setminus \Sigma(W)$

$$<W(\lambda)> <W(\lambda)^{-1}> = <W(\lambda)^{-1}> <W(\lambda)> = <I_Y> \ .$$

We shall now establish Properties (P_1) - (P_4) for the quintet $<\theta>$. Property (P_1) is clear, because $\sigma(<A>) \subset \sigma(A)$. Properties (P_2) and (P_3) follow from the identities

$$<W>(\lambda)<C>\{\lambda I_{<X>} - <A>\}^{-1} = <W(\lambda)C(\lambda-A)^{-1}> \ ;$$

$$<W>(\lambda)^{-1} - <C>\{\lambda I_{<X>} - <A>\}^{-1}<B> = <W(\lambda)^{-1} - C(\lambda-A)^{-1}B> \ ,$$

which hold true for $\lambda \in \Omega \setminus \Sigma(W)$.

It remains to establish Property (P_4). Choose $<(x_n)_{n=0}^{\infty}> \in <X>$, and let $<C><A>^k<(x_n)_{n=0}^{\infty}> = 0_{<Y>}$ $(k = 0,1,2,\ldots)$. Then

$$<C>\{\lambda I_{<X>} - <A>\}^{-1}<(x_n)_{n=0}^{\infty}> = 0_{<Y>} \ , \tag{2.3}$$

on a neighbourhood of infinity. Using Lemma 2.1 we see that Eq. (2.3) holds true for all $\lambda \notin \sigma(<A>)$, and therefore for all $\lambda \notin \sigma(A)$.

Since $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω, it follows from Corollary 1.3 that for some bounded Cauchy domain Δ such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$ we have

$$(2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1}d\lambda = I_X \ . \tag{2.4}$$

Observe that this integral is defined as a limit in the norm topology of $L(X)$ of a sequence of Riemann sums. Since $T \to \langle T \rangle$ is a continuous algebra homomorphism from $L(X)$ into $L(\langle X \rangle)$ which maps I_X into $I_{\langle X \rangle}$, it follows from Eq. (2.4) that

$$(2\pi i)^{-1} \int_{\partial\Delta} (\lambda - \langle A \rangle)^{-1} \langle B \rangle \langle W \rangle(\lambda) \langle C \rangle (\lambda - \langle A \rangle)^{-1} d\lambda = I_{\langle X \rangle} .$$

With the help of (2.3) we now obtain $\langle (x_n)_{n=0}^{\infty} \rangle = 0_{\langle Y \rangle}$, which establishes Property (P_4). □

THEOREM 2.8. *Let* $\theta = (A,B,C;X,Y)$ *be a spectral node for* W *on* Ω. *Then*

$$\theta^* = (A^*, C^*, B^*; X^*, C^*)$$

is a spectral node for W^* *on* Ω, *where* $(W^*)(\lambda) = W(\lambda)^*$, $\lambda \in \Omega$.

Proof. Properties (P_1) and (P_2) hold true by duality, and Property (P_3) is clear from Proposition 1.1. To prove (P_4), take $g \in \cap_{n=0}^{\infty} Ker\ B^*(A^*)^n$. Then $B^*(\lambda - A^*)^{-1} g$ vanishes on a neighbourhood of infinity. Lemma 2.1 implies that $B^*(\lambda - A^*)^{-1} g = 0$, $\lambda \notin \sigma(A^*) = \sigma(A)$. So for all $x \in X$ and $\lambda \notin \sigma(A)$ we have $g((\lambda - A)^{-1} Bx) = 0$.

Take $z \in X$. For the moment we fix $\lambda \in \Omega \setminus \sigma(A)$, put $x = W(\lambda) C (\lambda - A)^{-1} z$ and conclude that $g((\lambda - A)^{-1} BW(\lambda) C(\lambda - A)^{-1} z) = 0$. But then the latter identity holds for all $z \in X$ and $\lambda \in \Omega \setminus \sigma(A)$. We now apply Corollary 1.3 and infer that $g(z) = 0$ for all $z \in X$. Hence $g = 0$, which established Property (P_4). □

Note that Theorem 2.8 implies that for any spectral node

$$\overline{\text{span} \bigcup_{n=0}^{\infty} Im\ A^n B} = X .$$

It follows that, in the terminology of Chapter 3 of [3], spectral nodes are minimal nodes.

In the definition of a spectral node the roles of the operators B and C are not symmetric. Of course, in (P_1) and (P_2) they play analogous roles, but (P_3) and (P_4) are conditions on the pair (C,A) only. The analogues of (P_3) and (P_4) for the pair (A,B) are:

(P_3') The operator function $(\lambda - A)^{-1} BW(\lambda)$ has an analytic continuation to Ω ;

(P_4') $\overline{\text{span} \bigcup_{n=0}^{\infty} Im\ A^n B} = X$.

Note that from Proposition 1.1 and Theorem 2.8 we may conclude that a spectral node $\theta = (A,B,C;X,Y)$ for W on Ω has the properties (P_3') and (P_4'). Conversely, if the quintet $\theta = (A,B,C;X,Y)$ satisfies the conditions (P_1), (P_2), (P_3') and (P_4') for W on Ω, then θ is a spectral node for W on Ω. To see this, observe that $\theta^* = (A^*,B^*,C^*;X^*,Y^*)$ satisfies (P_1) to (P_4) for W^* on Ω, and hence θ^* is a spectral node for W^* on Ω. But then $\theta^{**} = (A^{**}, B^{**},C^{**};X^{**},Y^{**})$ is a spectral node for W^{**} on Ω, because of Theorem 2.8. It follows that conditions (P_3) and (P_4) hold true for the pair (C^{**},A^{**}), and thus, since X is embedded into X^{**}, the pair (C,A) has Properties (P_3) and (P_4) too. Consequently θ is a spectral node for W on Ω. Hence, the notion of a spectral node is completely symmetric with respect to B and C.

Suppose that $\theta = (A,B,C;X,Y)$ is a quintet such that for some open subset Ω of $\mathbb{C}$ and some analytic operator function $W\colon \Omega \to L(Y)$ with compact spectrum, the conditions (P_1), (P_2), (P_3) and (P_3') are fulfilled. Using the technique of the proof of Theorem 1.2 one easily shows that

$$\bigcap_{n=0}^{\infty} \mathit{Ker}\ CA^n \oplus \overline{\operatorname{span} \bigcup_{n=0}^{\infty} \mathit{Im}\ A^nB} = X ,$$

and that the projection of X onto the closed linear span of the set $\bigcup_{n=0}^{\infty} \mathit{Im}\ A^nB$ along $\bigcap_{n=0}^{\infty} \mathit{Ker}\ CA^n$ is given by the operator

$$(2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1}d\lambda ,$$

where Δ is a bounded Cauchy domain with $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. In that case (P_4) holds if and only if (P_4') holds.

3. Construction of Spectral Nodes and Linearization

Up to now several properties of spectral nodes have been established but not yet their existence. This will be done in the next theorem. In the next theorem we assume for simplicity that zero is inside Ω.

THEOREM 3.1. *Let $W\colon \Omega \to L(Y)$ be an analytic operator function with compact spectrum $\Sigma(W)$. Suppose that Δ is a bounded Cauchy domain containing* 0 *such that $\Sigma(W) \subset \Delta \subset \overline{\Delta} \subset \Omega$, and let* M *be the set of all continuous Y-valued functions f on the boundary $\partial\Delta$ which admit an analytic continuation to a Y-valued* *function in* $\mathbb{C}_\infty \setminus \Sigma(W)$ *vanishing at* ∞*, while* $W(\lambda)f(\lambda)$ *has an analytic continuation to* Ω*. The set* M *endowed with the supremum norm is a Banach space. Put*

$$A\colon M \to M\ ,\quad (Af)(z) = zf(z) - (2\pi i)^{-1}\int_{\partial\Delta} f(\omega)\,d\omega\ ;$$

$$B\colon Y \to M\ ,\quad (By)(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{W(w)^{-1}}{z-w}\, y\, dw\ ;$$

$$C\colon M \to Y\ ,\quad Cf = (2\pi i)^{-1}\int_{\partial\Delta} f(w)\,dw\ .$$

In the definition of B *the contour* Γ *is the boundary of a bounded Cauchy domain* Δ' *such that* $\Sigma(W) \subset \Delta' \subset \overline{\Delta'} \subset \Delta$*. Then* $(A,B,C;M,Y)$ *is a spectral node for* W *on* Ω*.*

From the proof of Theorem 3.1 it will be clear that this theorem remains valid if M is endowed with the L_2-norm (see the remark after the proof of Lemma 3.3). Hence we may conclude that for the case when Y is a separable Hilbert space the space on which the main operator of a spectral node acts may be taken to be a separable Hilbert space too.

The proof of Theorem 3.1 takes several steps. First we employ the calculus of spectral nodes, which has been developed in the previous section, to construct a spectral node for W on Ω assuming that a linearization of W on Ω is known. This will be done in Theorem 3.2 below. Next, we use the fact that explicit formulas for linearizations of W may be given. In fact, making use of the linearization formulas of [10], Section 2.2, we shall derive Theorem 3.1 as a consequence of Theorem 3.2.

THEOREM 3.2. *Consider on* Ω *the linearization*

$$W(\lambda) \oplus I_Z = E(\lambda)(\lambda - T)F(\lambda)\ ,\quad \lambda \in \Omega\ . \tag{3.1}$$

Let $\pi\colon Y \oplus Z \to Y$ *be the projection onto* Y *along* Z*, and let* τ *be the natural embedding of* Y *into* $Y \oplus Z$*. Further, let* Δ *be a bounded Cauchy domain in* Ω *such that* $\Sigma(W) \subset \Delta \subset \overline{\Delta} \subset \Omega$*. Put*

$$X = \operatorname{Im}\left[(2\pi i)^{-1}\int_{\partial\Delta} (\lambda - T)^{-1}\,d\lambda\right]\ ,$$

i.e., X *is the spectral subspace of* T *corresponding to the part of* $\sigma(T)$ *inside* Ω*. Define*

$$A\colon X \to X\ ,\quad A = T_{|X}\ ;$$

$$B\colon Y \to X\ , \quad B = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda - T)^{-1}E(\lambda)^{-1}\tau d\lambda\ ;$$

$$C\colon X \to Y\ , \quad C = (2\pi i)^{-1}\int_{\partial\Delta} \pi F(\lambda)^{-1}(\lambda - T)^{-1}d\lambda\ .$$

Then $(A,B,C;X,Y)$ is a spectral node for W on Ω.

Proof. By the linearization (3.1) it is clear that $\Sigma(W) = \sigma(T) \cap \Omega$. Let

$$P = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda - T)^{-1}d\lambda$$

be the Riesz projection of T corresponding to the part of $\sigma(T)$ inside Ω; so $X = Im\ P$. Further, let U be the space on which T acts, and let $\kappa\colon X \to U$ be the natural embedding of X into U. Then $(A,P,\kappa;X,U)$ is a spectral node for $\lambda - T$ on Ω (here P is considered as an operator from U into X).

Choose a fixed point $\lambda_0 \in \Omega$, and put $\tilde{E}(\lambda) = E(\lambda_0)^{-1}E(\lambda)$ and $\tilde{F}(\lambda) = F(\lambda)F(\lambda_0)^{-1}$. Then $\tilde{E}(\lambda)$, $\tilde{F}(\lambda)\colon U \to U$ are invertible and depend analytically on λ. Consider the operators $\tilde{B}\colon U \to X$ and $\tilde{C}\colon X \to U$, defined by

$$\tilde{B} = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda - A)^{-1}P\tilde{E}(\lambda)^{-1}d\lambda = (2\pi i)^{-1}\int_{\partial\Delta} (\lambda - T)^{-1}\tilde{E}(\lambda)^{-1}d\lambda\ ,$$

$$\tilde{C} = (2\pi i)^{-1}\int_{\partial\Delta} \tilde{F}(\lambda)^{-1}(\lambda - A)^{-1}\kappa\, d\lambda = (2\pi i)^{-1}\int_{\partial\Delta} \tilde{F}(\lambda)^{-1}(\lambda - T)^{-1}d\lambda.$$

By applying Theorem 2.3, we get a spectral node on Ω for the operator function V, defined by

$$V(\lambda) = \tilde{E}(\lambda)(\lambda - T)\tilde{F}(\lambda)\ , \qquad \lambda \in \Omega\ ,$$

namely the quintet $(A,\tilde{B},\tilde{C};X,U)$. Note that $W(\lambda) \oplus I_Z = E(\lambda_0)V(\lambda)\cdot F(\lambda_0)$ for each $\lambda \in \Omega$. It follows that

$$(A,\tilde{B}E(\lambda_0)^{-1},\ F(\lambda_0)^{-1}\tilde{C};X,Y \oplus Z)$$

is a spectral node for $W(\lambda) \oplus I_Z$ on Ω, and hence

$$(A,\tilde{B}E(\lambda_0)^{-1}\tau,\pi F(\lambda_0)^{-1}\tilde{C};X,Y) \tag{3.2}$$

is a spectral node for W on Ω (cf. Theorem 2.5). Finally, observe that $B = \tilde{B}E(\lambda_0)^{-1}\tau$ and $C = \pi\cdot F(\lambda_0)^{-1}\tilde{C}$. □

We shall use Theorem 3.2 to derive Theorem 3.1. We begin with a remark. Let Ω_0 be an open set in $\mathbb{C}$ such that $\Sigma(W) \subset \Omega_0 \subset \Omega$. If $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω_0, then trivially θ is a spectral node for W on Ω. Hence it suffices to construct a spectral node for W on some open neighbourhood of $\Sigma(W)$. Now choose $\Omega_0 = \Delta$, where Δ is as in Theorem 3.1. We shall apply the Gohberg-Kaashoek-Lay linearization of W on Δ (cf. [10], Section 2.2) and Theorem 3.2 to construct a spectral node for W on Δ, and hence for W on Ω.

Let $C(\partial\Delta,Y)$ be the Banach space of all continuous Y-valued functions on $\partial\Delta$ endowed with the supremum norm, and let T be the operator on $C(\partial\Delta,Y)$, defined by

$$(Tf)(z) = zf(z) + (2\pi i)^{-1}\int_{\partial\Delta} [W(\lambda) - I]f(\lambda)d\lambda \ . \tag{3.3}$$

The operator T is a linearization of W on Δ (cf. [10], Theorem 2.2). In fact, we have (cf. [10], Section 2.4)

$$G(\lambda)[W(\lambda) \oplus I_Z] = (\lambda-T)F(\lambda) \ , \quad \lambda \in \Delta \ , \tag{3.4}$$

where $Z = \{g \in C(\partial\Delta,Y) : (2\pi i)^{-1}\int_{\partial\Delta} z^{-1}g(z)dz = 0\}$ and

$$(G(\lambda)(y,g))(z) = -y - g(z) - \frac{1}{2\pi i}\int_{\partial\Delta} \frac{W(w)-I}{w-\lambda}(y+g(w))dw \ ; \tag{3.5}$$

$$(F(\lambda)(y,g))(z) = (z-\lambda)^{-1}(y+g(z)) \ . \tag{3.6}$$

The operators $G(\lambda)$, $F(\lambda)$: $Y \oplus Z \to C(\partial\Delta,Y)$ are invertible and depend analytically on λ. It is known (cf. [10], Theorem 2.3) that

$$\sigma(T) = \Sigma(W) \cup \partial\Delta \ .$$

To construct from this linearization a spectral node we have to identify the spectral subspace of T corresponding to the part of $\sigma(T)$ inside Δ. This is done in the next lemma.

LEMMA 3.3. *The spectral subspace M of T corresponding to the part of $\sigma(T)$ inside Δ consists of all $f \in C(\partial\Delta,Y)$ that can be extended to a Y-valued function analytic outside $\Sigma(W)$ and vanishing at ∞, while Wf has an analytic continuation to Ω. Further, for each $f \in M$*

$$(Tf)(z) = zf(z) - (2\pi i)^{-1}\int_{\partial\Delta} f(w)dw \ . \tag{3.7}$$

<u>Proof</u>. For each $f \in C(\partial\Delta,Y)$ and $\lambda \in \Delta \setminus \Sigma(W)$ we have

$$[(\lambda-T)^{-1}f](z) = \frac{f(z)}{\lambda - z} - \frac{W(\lambda)^{-1}}{\lambda - z}\frac{1}{2\pi i}\int_{\partial\Delta} \frac{W(w)-I}{w-\lambda} f(w)dw \ . \tag{3.8}$$

Let Δ' be a bounded Cauchy domain such that $\Sigma(W) \subset \Delta' \subset \overline{\Delta'} \subset \Delta$, and put $\Gamma = \partial\Delta'$. The Riesz projection P of T corresponding to the part of $\sigma(T)$ inside Δ is then given by

$$(3.9)\qquad (Pf)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)^{-1}}{z-\lambda}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(w)-I}{w-\lambda}f(w)\,dw\right)d\lambda\ .$$

First assume that f can be extended to a Y-valued function analytic outside $\Sigma(W)$ and vanishing at ∞, while Wf has an analytic continuation to Ω. Then formula (3.9) implies that for all $z \in \partial\Delta$ we have $(Pf)(z) = f(z)$. So $f \in M$. Also, in this case,

$$\int_{\partial\Delta} W(\lambda)f(\lambda)\,d\lambda = 0\ .$$

By substituting this into the definition of T, we see that (3.7) holds true.

Secondly, assume that $f \in M$. Then $f = Pf$, and hence we can apply formula (3.9) to show that f can be extended to a function analytic outside Δ' and vanishing at ∞. This extension of f will also be denoted by f. Take $z_0 \in \Delta \setminus \overline{\Delta'}$. To show that Wf admits an analytic continuation to Ω, it suffices to show the equality

$$(3.10)\qquad \frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)f(z)}{z-z_0}\,dz = W(z_0)f(z_0)\ .$$

Using $f = Pf$, we have

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)}{z-z_0}\left\{\frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)^{-1}}{z-\lambda}\cdot\right.$$
$$\left.\cdot\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(w)-I}{w-\lambda}f(w)\,dw\right)d\lambda\right\}dz\ .$$

As the integrand is a continuous function in (z,λ,w) on the compact set $\partial\Delta \times \Gamma \times \partial\Delta$, we may apply Fubini's theorem. At first, we evaluate the integral over z and obtain

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)}{(z-z_0)(z-\lambda)}\,dz = \frac{W(\lambda)-W(z_0)}{\lambda - z_0}\ .$$

Substitution yields

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)-W(z_0)}{\lambda-z_0}W(\lambda)^{-1}\cdot$$
$$\cdot\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(w)-I}{w-\lambda}f(w)\,dw\right)d\lambda\ .$$

Now we split the double integral at the right-hand side into two terms, apply Fubini's theorem to the first one of these terms, and obtain

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(z)f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\partial\Delta} W(w)f(w)\left(\frac{1}{2\pi i}\int_\Gamma \frac{d\lambda}{(\lambda-z_0)(w-\lambda)}\right)dw\ +$$
$$W(z_0)\ \frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)^{-1}}{z_0-\lambda}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{W(w)-I}{w-\lambda}\,f(w)\,dw\right)d\lambda\ .$$

As z_0 and w are lying outside $\overline{\Delta'}$, the integral over λ appearing in the first term at the right-hand side vanishes. Comparing the second term at the right-hand side to formula (3.9), we now obtain Eq. (3.10). □

In the case when Y is a separable Hilbert space, none of the previous arguments is affected if we take the space $L_2(\partial\Delta, Y)$ of strongly measurable Y-valued L_2-functions on $\partial\Delta$ instead of $C(\partial\Delta, Y)$. For this case, and Δ the unit disc, Lemma 3.3 has been established in [23], where Fourier expansion was used. In this particular case the operator (3.7) coincides with the linearizator introduced in [20], where the space M has been defined by using the characterization described in Lemma 3.3. It was shown in [20] that $\sigma(T_{|M}) = \Sigma(W)$; this fact is now immediate from Lemma 3.3 and the linearization (3.3). In the present form Lemma 3.3 also appeared in [21].

Let us finally prove Theorem 3.1.

Proof of Theorem 3.1. Let T be the operator defined by (3.3). We know that T is a linearization of W on Δ. So we may apply Theorem 3.1 to the linearization (3.4). By Lemma 3.3 the spectral subspace of T corresponding to the part of $\sigma(T)$ inside Δ coincides with M and

$$A = T_{|M}$$

(cf. (3.7)). Let be the boundary of a bounded Cauchy domain Δ' such that $\Sigma(W) \subset \Delta' \subset \overline{\Delta'} \subset \Delta$. To finish the proof it suffices to show that the operators B and C introduced in the theorem satisfy the following identities:

(3.11) $$B = (2\pi i)^{-1}\int_\Gamma (\lambda - T)^{-1} G(\lambda)\tau\, d\lambda\ ;$$

$$(3.12)\qquad C = (2\pi i)^{-1}\int_\Gamma \pi F(\lambda)^{-1}(\lambda - T)^{-1}d\lambda ,$$

where $F(\lambda)$ and $G(\lambda)$ are given by Eqs. (3.5) and (3.6), the map π is the projection of $Y \oplus Z$ onto Y along Z and τ is the natural embedding of Y into $Y \oplus Z$. Recall that $Z = \{f \in C(\partial\Delta, Y):$ $(2\pi i)^{-1}\int_{\partial\Delta} z^{-1}f(z)dz = 0\}$. Then $(\tau y)(z) = y(z \in \partial\Delta)$ and $\pi f =$ $(2\pi i)^{-1}\int_{\partial\Delta} z^{-1}f(z)dz$.

To compute the right-hand side of (3.11), note that

$$(\lambda - T)^{-1}G(\lambda)\tau y = F(\lambda)(W(\lambda)^{-1}y, 0).$$

Using the definition of $F(\lambda)$ (see (3.5)), we obtain for each $z \in \partial\Delta$ and $y \in Y$ the following equality:

$$\left(\frac{1}{2\pi i}\int_\Gamma (\lambda - T)^{-1}G(\lambda)\tau y d\lambda\right)dz = \frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)^{-1}}{z - \lambda}\,d\lambda.$$

This proves (3.11).

Take $f \in M$. Denote by f also the analytic continuation of f to $\mathbb{C}_\infty \setminus \Sigma(W)$. Since Wf has an analytic continuation to Ω, we see from (3.7) that

$$[(\lambda - T)^{-1}f](z) = \frac{f(z) - f(\lambda)}{\lambda - z} .$$

Thus

$$\pi F(\lambda)^{-1}(\lambda - T)^{-1}f = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(\lambda) - f(z)}{z}\,dz = f(\lambda) ,$$

and (3.12) holds. □

We now state a corollary that is immediate from Theorem 3.1, the linearization (3.4) and Theorem 1.2.

COROLLARY 3.4. *The spectrum of the main operator of a spectral node for W on Ω coincides with $\Sigma(W)$.*

From Theorem 3.2 it follows that on a bounded open set Ω every analytic operator function W with compact spectrum has a spectral linearization (see the introduction), namely, the main operator of a spectral node for W on Ω. To see this, recall that on a bounded open set Ω every analytic operator function W has a linearization T (cf. [22, 5]), i.e., there exist a Banach space Z and invertible operators $E(\lambda)$ and $F(\lambda)$ depending analytically on $\lambda \in \Omega$ such that

$$(3.13)\qquad W(\lambda) \oplus I_Z = E(\lambda)(\lambda - T)F(\lambda) , \quad \lambda \in \Omega .$$

From this equation and the compactness of $\Sigma(W)$ it is clear that $\sigma(T) \cap \Omega = \Sigma(W)$, while $\Sigma(W)$ and $\sigma(T) \setminus \Omega$ are spectral subsets of T. So we may write $T = T_0 \oplus T_1$, where $\sigma(T_0) = \sigma(T) \cap \Omega$ and $\sigma(T_1) = \sigma(T) \setminus \Omega$. If X and U are the spaces on which the operators T_0 and T_1 act, respectively, then it is clear from (3.13) that

$$W(\lambda) \oplus I_Z = E(\lambda)[(\lambda-T_0) \oplus I_U]\tilde{F}(\lambda) \quad , \quad \lambda \in \Omega ,$$

where $\tilde{F}(\lambda) = [I_X \oplus (\lambda-T_1)]\, F(\lambda)$ is an invertible operator depending analytically on $\lambda \in \Omega$. So the operator T_0 is a spectral linearization for W on Ω. From Theorem 3.2 it follows that T_0 is the main operator of a spectral node for W on Ω. Hence on a bounded domain Ω the main operator of some (and, by Theorem 1.2, every) spectral node for W on Ω is a spectral linearization for W on Ω.

We shall see in Section 4 that a spectral linearization for W on Ω exists also if Ω is unbounded.

We conclude this section with a remark concerning functions with continuous and invertible boundary values. Let Ω be a bounded Cauchy domain, and let $W\colon \overline{\Omega} \to L(Y)$ be an operator function that is analytic and continuous up to the boundary of Ω. Assume that W has invertible boundary values. For such a function the spectrum $\Sigma(W) = \{\lambda \in \overline{\Omega}\colon W(\lambda) \text{ is not invertible}\}$ is a compact subset of Ω. The definition of a spectral node applies to W without essential changes, and all the results and constructions of Sections 1 - 3 remain valid. Of course, in all the considerations one can choose the bounded Cauchy domain Δ to be Ω itself.

4. Characterizations of Spectral Nodes and Inverse Problems

The main theorem of this section is the following:

THEOREM 4.1. *Let $A\colon X \to X$, $B\colon Y \to X$ and $C\colon X \to Y$ be given operators. Then $\theta = (A,B,C;X,Y)$ is a spectral node for some analytic operator function $W\colon \Omega \to L(Y)$ with compact spectrum if and only if the following two conditions hold:*

(i) $\sigma(A) \subset \Omega$;

(ii) *there exists an analytic operator function $H\colon \Omega \to L(Y)$ such that*

$$(4.1) \qquad E(\lambda) = \begin{bmatrix} A-\lambda & B \\ C & H(\lambda) \end{bmatrix}$$

is invertible for all $\lambda \in \Omega$.

More precisely, if (i) and (ii) hold, then $W: \Omega \to L(Y)$, *defined by*

$$(4.2) \qquad E(\lambda)^{-1} = \begin{bmatrix} * & * \\ * & W(\lambda) \end{bmatrix}, \quad \lambda \in \Omega ,$$

is analytic on Ω, *has a compact spectrum,* $\theta = (A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *and*

$$(4.3) \qquad W(\lambda)^{-1} = H(\lambda) + C(\lambda-A)^{-1}B , \quad \lambda \in \Omega \setminus \sigma(A) .$$

Furthermore, W *is uniquely determined by these properties.*

Conversely, if $\theta = (A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *then (i), (ii) and* (4.2) *hold for a unique operator function* H *defined by* (4.3).

Proof. Suppose that conditions (i) and (ii) hold. For $\lambda \in \Omega$ let $W(\lambda)$ be defined by the right lower entry of the block matrix $E(\lambda)^{-1}$ (cf. (4.2)). We shall show that $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω.

Write

$$E(\lambda)^{-1} = \begin{bmatrix} T_{11}(\lambda) & T_{12}(\lambda) \\ T_{21}(\lambda) & W(\lambda) \end{bmatrix}, \quad \lambda \in \Omega ,$$

where $T_{11}(\lambda) \in L(X)$, $T_{12}(\lambda) \in L(Y,X)$ and $T_{21}(\lambda) \in L(X,Y)$ depend analytically on λ. From the identities $E(\lambda)E(\lambda)^{-1} = I$ and $E(\lambda)^{-1}E(\lambda) = I$ we have for all $\lambda \in \Omega$:

$$(A-\lambda)T_{12}(\lambda) = -BW(\lambda) ; \qquad T_{21}(\lambda)(A-\lambda) = -W(\lambda)C ;$$

$$CT_{12}(\lambda) + H(\lambda)W(\lambda) = I; \qquad T_{21}(\lambda)B + W(\lambda)H(\lambda) = I ;$$

$$T_{11}(\lambda)(A-\lambda) + T_{12}(\nu)C = I .$$

From these equations we easily compute that for $\lambda \in \Omega \setminus \sigma(A)$ we have

$$T_{12}(\lambda) = (\lambda-A)^{-1}BW(\lambda) ; \qquad T_{21}(\lambda) = W(\lambda)C(\lambda-A)^{-1} ;$$

$$[H(\lambda) + C(\lambda-A)^{-1}B]W(\lambda) = I; \quad W(\lambda)[H(\lambda) + C(\lambda-A)^{-1}B] = I ;$$

$$T_{11}(\lambda) = -(\lambda-A)^{-1} + (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1} .$$

We know that $\sigma(A) \subset \Omega$, which establishes (P_1). From the above equations it appears that Properties (P_3) and $(P_3{}')$ hold true (see at the end of Section 2 for $(P_3{}')$). Also the operator function W has a compact spectrum contained in $\sigma(A)$, while

$$W(\lambda)^{-1} = H(\lambda) + C(\lambda-A)^{-1}B , \quad \lambda \in \Omega \setminus \sigma(A) .$$

From this identity Property (P_2) is clear.

To establish (P_4), we choose a bounded Cauchy domain Δ such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. From (4.3) and the analyticity of T_{11} on Ω it appears that

$$(2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1}d\lambda = I .$$

In view of the remark made in the last paragraph of Section 2, Property (P_4) follows. Hence $\theta = (A,B,C;X,Y)$ is, indeed, a spectral node for W on Ω.

Conversely, let $\theta = (A,B,C;X,Y)$ be a spectral node for W on Ω, and use Property (P_2) to define an analytic operator function H: $\Omega \to L(Y)$ by Eq. (4.3). Define Z: $\Omega \setminus \sigma(A) \to L(X)$ by

$$Z(\lambda) = -(\lambda-A)^{-1} + (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1} . \tag{4.4}$$

First we show that Z has an analytic continuation to Ω.

Let Δ be a bounded Cauchy domain such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$, and take $\mu \in \Omega \setminus \overline{\Delta}$. With the help of the identity

$$(\mu-\lambda)^{-1}(\lambda-A)^{-1} = (\mu-A)^{-1}(\lambda-A)^{-1} + (\mu-\lambda)^{-1}(\mu-A)^{-1}(\lambda \in \partial\Delta),$$

we easily compute that

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{Z(\lambda)}{\mu-\lambda}\, d\lambda = -(\mu-A)^{-1} + (\mu-A)^{-1}\cdot \frac{1}{2\pi i} \int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)C(\lambda-A)^{-1}d\lambda +$$
$$(\mu-A)^{-1}\cdot \frac{1}{2\pi i} \int_{\partial\Delta} \frac{BW(\lambda)C(\lambda-A)^{-1}}{\mu-\lambda}\, d\lambda = 0$$

in view of Corollary 1.3, Property (P_3) and the fact that $\mu \notin \overline{\Delta}$. So Z has an analytic continuation to Ω.

From Property (P_3) and Proposition 1.1 it is clear that

$$\begin{bmatrix} Z(\lambda) & (\lambda-A)^{-1}BW(\lambda) \\ W(\lambda)C(\lambda-A)^{-1} & W(\lambda) \end{bmatrix} \tag{4.5}$$

has an analytic continuation to Ω. We define analytic operator functions $H\colon \Omega \to L(Y)$ by (4.3), and $E\colon \Omega \to L(X \oplus Y)$ by (4.1). Then a straightforward computation shows that for $\lambda \in \Omega \setminus \sigma(A)$ the operator $E(\lambda)$ is invertible and that its inverse is given by (4.5). By analytic continuation, $E(\lambda)$ is invertible for all $\lambda \in \Omega$, which establishes Condition (ii). Condition (i) is identical to (P_1). Finally, the uniqueness of $H(\lambda)$ follows from (4.3). □

Note that in condition (ii) of Theorem 4.1 the compactness of the spectrum of the operator function W does not appear. This fact we shall exploit later in part III of the paper.

COROLLARY 4.2. *If $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω, then the operator A is a spectral linearization of W on Ω. In fact,*

$$(4.6)\qquad E(\lambda)\begin{bmatrix} I_X & 0 \\ 0 & W(\lambda)\end{bmatrix} = \begin{bmatrix} \lambda - A & 0 \\ 0 & I_Y \end{bmatrix} F(\lambda)\ , \qquad (\lambda \in \Omega)\ ,$$

where $E(\lambda)$ and $F(\lambda)$ are invertible operators depending analytically on the parameter $\lambda \in \Omega$ and are given by

$$E(\lambda) = \begin{bmatrix} A-\lambda & B \\ C & H(\lambda)\end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} -I & (\lambda-A)^{-1}BW(\lambda) \\ C & I-C(\lambda-A)^{-1}BW(\lambda)\end{bmatrix}.$$

Here $H(\lambda) = W(\lambda)^{-1} - C(\lambda-A)^{-1}B$.

Proof. The invertibility of $E(\lambda)$ for all $\lambda \in \Omega$ is clear from the previous theorem. The inverse of $F(\lambda)$ is easy to calculate and is given by

$$F(\lambda)^{-1} = \begin{bmatrix} -I+(\lambda-A)^{-1}BW(\lambda)C & (\lambda-A)^{-1}BW(\lambda) \\ C & I \end{bmatrix}.$$

Finally, Eq. (4.6) is established directly. □

From Property (P_2) it is clear that $\Sigma(W) \subset \sigma(A)$ if A is the main operator of a spectral node for W on Ω. Using Corollary 4.2 we directly infer that

$$\Sigma(W) = \sigma(A)\ .$$

From Corollary 4.2 it is also clear that any analytic operator function with compact spectrum has a spectral linearization (note that in the previous section we only proved this for operator functions on a bounded domain). The converse of Corollary 4.2, namely that any spectral linearization of W on Ω is the main operator of a spectral node for W on Ω, will be proved in the next section.

We remark that Theorem 4.1 and Corollary 4.2 have obvious analogues for analytic operator functions on a bounded Cauchy domain Ω with continuous and invertible boundary values. For example, if $W\colon \overline{\Omega} \to L(Y)$ is an analytic (in Ω) operator function with continuous and invertible boundary values, and $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω, then there exists an analytic operator function $H\colon \overline{\Omega} \to L(Y)$ with continuous boundary values such that the operator $E(\lambda)$, defined by (4.1), is invertible for all $\lambda \in \overline{\Omega}$. Further, the spectral linearization (4.6) holds on the closure of Ω.

Given an operator function $W\colon \overline{\Omega} \to L(Y)$ as in the preceding paragraph, one can find an analytic (in Ω) operator function $\tilde{W}\colon \overline{\Omega} \to L(Y)$ with continuous and invertible boundary values, which satisfies the following properties:

(a) $\tilde{W}^{-1}$ has an extension to $\mathbb{C}_\infty \setminus \Sigma(W)$ that is analytic except possibly for a finite number of poles outside $\overline{\Omega}$;

(b) $\tilde{W}$ and W have the same spectral nodes.

Moreover, if $\mathbb{C} \setminus \Omega$ is connected, then we can choose $\tilde{W}$ in such a way that $\tilde{W}^{-1}$ is analytic in $\mathbb{C}_\infty \setminus \Sigma(W)$ except possibly for a pole at infinity. To prove this one applies the operator version of Runge's theorem (cf. [7], Lemma 1.1 in [15]) to approximate on $\overline{\Omega}$ the operator function $H(\lambda)$ appearing in (4.1) by a rational operator function $R(\lambda)$ with poles in $\mathbb{C}_\infty \setminus \overline{\Omega}$. If Ω has a connected complement, then $R(\lambda)$ may be chosen to be a polynomial.

As a final remark in connection with Theorem 4.1, we consider the special class of monic operator polynomials. Let $L(\lambda)$ be a monic operator polynomial on Y, and let $(A,B,C;X,Y)$ be a spectral node for L on the full complex plane. Then it follows from Theorem 4.1 and the identity $L(\lambda)^{-1} = C(\lambda-A)^{-1}B$ $(\lambda \in \mathbb{C} \setminus \Sigma(L))$ that the operator

$$\text{(4.7)} \qquad \begin{bmatrix} A-\lambda & B \\ C & 0 \end{bmatrix}$$

is invertible for all $\lambda \in \mathbb{C}$, and its inverse is given by

$$\begin{bmatrix} A(\lambda) & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} Z(\lambda) & (\lambda-A)^{-1}BL(\lambda) \\ L(\lambda)C(\lambda-A)^{-1} & L(\lambda) \end{bmatrix},$$

where

$$Z(\lambda) = -(\lambda-A)^{-1} + (\lambda-A)^{-1}BL(\lambda)C(\lambda-A)^{-1} .$$

For the spectral node

$$\left(K,\ [\,0\ \ldots\ 0\quad I\,],\ \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix};\ Y^{\ell},\ Y \right)$$

of L, where ℓ is the degree of L and K is its companion operator (see formula (0.2) and [14]), the invertibility of (4.7) can be seen directly, because in this case (4.7) is equal to

$$\text{(4.8)} \qquad \begin{bmatrix} -\lambda & I & 0 & \cdots & & & 0 \\ 0 & -\lambda & I & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\lambda & I & & 0 \\ -A_0 & -A_1 & \cdots & -A_{\ell-2} & -\lambda-A_{\ell-1} & & I \\ I & 0 & \cdots & 0 & 0 & & 0 \end{bmatrix},$$

where A_j, $j = 0,\ldots,\ell-1$, are the coefficients of L.

The characterization of spectral nodes, given by Theorem 4.1, can be viewed as a solution of an inverse problem. Let $A\colon X \to X$, $B\colon Y \to X$ and $C\colon X \to Y$ be given bounded linear operators, and let Ω be an open set in $\mathbb{C}$ containing $\sigma(A)$. The inverse problem concerns the following question: Under what conditions on A, B and C is the quintet $(A,B,C;X,Y)$ a spectral node for an analytic operator function W on Ω? If one specifies the class of functions, then more explicit solutions of this inverse problem than the conditions of Theorem 4.1 are known. For example (cf. [12, 2]), the quintet $(A,B,C;X,Y)$ is a spectral node for a monic operator polynomial on the full complex plane if and only if for some positive integer ℓ the map

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\ell-1} \end{bmatrix} : \quad X \to Y^{\ell}$$

is invertible and its inverse is of the form $[\ * \ \ldots \ * \quad B\ ]$.

The next theorem solves the inverse problem for another class of operator functions.

THEOREM 4.3. *Let* $A\colon X \to X$, $B\colon Y \to X$ *and* $C\colon X \to Y$ *be given operators. Then* $\theta = (A,B,C;X,Y)$ *is a spectral node for some analytic operator function* $W\colon \Omega \to L(Y)$ *with compact spectrum and with the property that* $W(\lambda)^{-1}$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \Sigma(W)$ *if and only if the following two conditions hold:*

(i) $\sigma(A) \subset \Omega$;

(ii) *there exists an operator* $D\colon Y \to Y$ *such that*

$$E(\lambda) = \begin{bmatrix} A-\lambda & B \\ C & D \end{bmatrix}$$

is invertible for all $\lambda \in \Omega$.

More precisely, if (i) and (ii) hold, then $W\colon \Omega \to L(Y)$, *defined by*

$$E(\lambda)^{-1} = \begin{bmatrix} * & * \\ * & W(\lambda) \end{bmatrix}, \quad \lambda \in \Omega ,$$

has the required properties.

Conversely, if $\theta = (A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *and* $W(\lambda)^{-1}$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \Sigma(W)$, *then (i) and (ii) hold for a unique operator* D *defined by*

(4.10) $$D = \lim_{\lambda\to\infty} W(\lambda)^{-1} .$$

Furthermore, W *is uniquely determined by these properties and*

(4.11) $$W(\lambda)^{-1} = D + C(\lambda-A)^{-1}B , \quad \lambda \in \Omega \setminus \sigma(A) .$$

Finally, if $\lambda_0 \in \Omega$ *and*

$$(4.12)\qquad E(\lambda_0)^{-1} = \begin{bmatrix} S & Q \\ R & U \end{bmatrix},$$

then W is also given by

$$(4.13)\qquad W(\lambda) = U + (\lambda-\lambda_0)R\big(I - (\lambda-\lambda_0)S\big)^{-1}Q\,, \quad \lambda \in \Omega\,.$$

<u>Proof</u>. Suppose that Conditions (i) and (ii) hold. For $\lambda \in \Omega$ let $W(\lambda)$ be defined by the right lower entry of the block matrix $E(\lambda)^{-1}$. Then it is clear from Theorem 4.1 that $W\colon \Omega \to L(Y)$ is an analytic operator function with compact spectrum such that $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω. Using Eq. (4.3) (with $H(\lambda) = D$) we obtain (4.11). So $W(\lambda)^{-1}$ has an analytic continuation to $\mathbb{C}_\infty \setminus \Sigma(W)$.

Conversely, let $\theta = (A,B,C;X,Y)$ be a spectral node for W on Ω, and suppose that $W(\lambda)^{-1}$ has an analytic continuation to $\mathbb{C}_\infty \setminus \Sigma(W)$. Then it is clear from Theorem 4.1 that there exists an analytic operator function $H\colon \Omega \to L(Y)$ such that the operator $E(\lambda)$, defined by (4.1), is invertible for all $\lambda \in \Omega$. Further, we exploit Eq. (4.3) and see that

$$H(\lambda) = W(\lambda)^{-1} - C(\lambda-A)^{-1}B, \quad \lambda \in \Omega \setminus \Sigma(W)\;.$$

Liouville's theorem yields that $H(\lambda) \equiv D$ is constant. But then Conditions (i) and (ii) and Eq. (4.10) are clear.

It remains to establish Eq. (4.13). Let $\lambda_0 \in \Omega$, and let $E(\lambda_0)^{-1}$ be given by (4.12). Writing $A - \lambda = (A-\lambda_0) - (\lambda-\lambda_0)$ and inserting (4.12) we get

$$\begin{bmatrix} A-\lambda_0 & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} A-\lambda & B \\ C & D \end{bmatrix} = \begin{bmatrix} I-(\lambda-\lambda_0)S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -(\lambda-\lambda_0)R & I \end{bmatrix}.$$

Since for all $\lambda \in \Omega$ the operator $E(\lambda)$ is invertible, it follows from the above identity that $I - (\lambda-\lambda_0)S$ is invertible for all $\lambda \in \Omega$, while

$$E(\lambda)^{-1} = \begin{bmatrix} I & 0 \\ (\lambda-\lambda_0)R & I \end{bmatrix} \begin{bmatrix} (I-(\lambda-\lambda_0)S)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S & Q \\ R & U \end{bmatrix},$$

and therefore

$$(4.14)\quad E(\lambda)^{-1} = \begin{bmatrix} S(I-(\lambda-\lambda_0)S)^{-1} & (I-(\lambda-\lambda_0)S)^{-1}Q \\ R(I\ (\lambda-\lambda_0)S)^{-1} & U+(\lambda-\lambda_0)R(I-(\lambda-\lambda_0)S)^{-1}Q \end{bmatrix}.$$

But according to Theorem 4.1 the right lower entry of the block matrix $E(\lambda)^{-1}$ coincides with $W(\lambda)$, $\lambda \in \Omega$, which establishes Eq. (4.13). □

Observe that the above argument proves the following general statement: Let $A \in L(X)$, $B \in L(Y,X)$, $C \in L(X,Y)$ and $D \in L(Y)$ be operators. Then the operator

$$E(\lambda) = \begin{bmatrix} A-\lambda & B \\ C & D \end{bmatrix}$$

is invertible for all $\lambda \in \Omega$ if and only if $E(\lambda_0)$ is invertible for some $\lambda_0 \in \mathbb{C}$ and $I - (\lambda-\lambda_0)S$ is invertible for all $\lambda \in \Omega$, where S is defined by the equality

$$E(\lambda_0)^{-1} = \begin{bmatrix} S & * \\ * & * \end{bmatrix}.$$

To conclude this section we remark that an analogue of Theorem 4.3 holds for the case when Ω is a bounded Cauchy domain and W is an analytic operator function with continuous and invertible boundary values.

5. Spectral Linearization and Equivalence

In this section we show in general that a spectral linearization of an analytic operator function is the main operator of a spectral node and conversely. Using this result we prove that two analytic operator functions have similar spectral linearizations if and only if they are equivalent after some extension, and we specify the extension spaces.

We recall the definition of a spectral linearization (see Introduction). Given an analytic operator function $W\colon \Omega \to L(Y)$ with compact spectrum, then an operator $A \in L(X)$ is called a *spectral linearization* of W on Ω if $\sigma(A) \subset \Omega$ and there exist Banach spaces Z and U, and invertible operators $E(\lambda) \in L(X \oplus U, Y \oplus Z)$ and $F(\lambda) \in L(Y \oplus Z, X \oplus U)$ depending analytically on $\lambda \in \Omega$, such that

$$(5.1) \qquad W(\lambda) \oplus I_Z = E(\lambda)[(\lambda-A) \oplus I_U]F(\lambda) \ , \quad \lambda \in \Omega \ .$$

In the previous section (cf. Corollary 4.2) we have shown that the main operator of a spectral node for W on Ω is a spectral linearization of W on Ω. The next theorem shows the converse to be true, and hence by Corollary 4.2 we can always take $Z = X$ and $U = Y$ in (5.1).

THEOREM 5.1. *A spectral linearization of W on Ω is the main operator of a spectral node for W on Ω, and conversely. In particular, a spectral linearization of W is uniquely determined up to similarity.*

Proof. Let $A \in L(X)$ be a spectral linearization of W on Ω, and suppose that the linearization is given by Eq. (5.1). Clearly, $(A,I_X,I_X;X,X)$ is a spectral node for $\lambda-A$ on Ω, because $\sigma(A) \subset \Omega$. With the help of Theorem 2.5 it is clear that $(A,\pi_X,\tau_X;X,X\oplus U)$ is a spectral node for $(\lambda-A) \oplus I_U$ on Ω. Here π_X: $X \oplus U \to X$ is the projection of $X \oplus U$ on X along U, and τ_X: $X \to X \oplus U$ is the natural embedding of X into $X \oplus U$. Next, we apply Theorem 2.3 and conclude that $(A,R_E,Q_F;X,Y\oplus Z)$ is a spectral node for $W(\cdot) \oplus I_Z$ on Ω, where

$$R_E = (2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}\pi_X E(\lambda)^{-1} d\lambda \ ,$$

$$Q_F = (2\pi i)^{-1} \int_{\partial\Delta} F(\lambda)^{-1}\tau_X(\lambda-A)^{-1} d\lambda \ ,$$

and Δ is a bounded Cauchy domain such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. Finally, we apply Theorem 2.5 again and infer that $(A,B,C;X,Y)$ is a spectral node for W on Ω. Here

$$B = (2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}\pi_X E(\lambda)^{-1}\tau_Y d\lambda \ ,$$

$$C = (2\pi i)^{-1} \int_{\partial\Delta} \pi_Y F(\lambda)^{-1}\tau_X(\lambda-A)^{-1} d\lambda \ ,$$

π_Y: $Y \oplus Z \to Y$ is the projection of $Y \oplus Z$ onto Y along Z, and τ_Y: $Y \to Y \oplus Z$ is the natural embedding of Y into $Y \oplus Z$. Hence the spectral linearization A is, indeed, the main operator of a spectral node for W on Ω.

The converse part of this theorem has been derived in Section 4 as a consequence of Corollary 4.2. The uniqueness up to similarity of a spectral linearization of W on Ω is now clear from Theorem 1.2. □

The next result describes the similarity referred to in Theorem 5.1.

THEOREM 5.2. *Let A_1 and A_2 be operators acting on the Banach spaces X_1 and X_2 respectively. Suppose that Z_1 and Z_2 are Banach spaces such that $(\lambda-A_1)\oplus I_{Z_1}$ and $(\lambda-A_2)\oplus I_{Z_2}$ are equivalent on an open set Ω containing $\sigma(A_1)\cup\sigma(A_2)$. Then A_1 and A_2 are similar. In fact, if the equivalence is given by*

$$(\lambda-A_1)\oplus I_{Z_1} = E(\lambda)[(\lambda-A_2)\oplus I_{Z_2}]F(\lambda),\ \lambda\in\Omega,$$

then $SA_1 = A_2S$, where $S : X_1 \to X_2$ is an invertible operator defined by

$$S = (2\pi i)^{-1}\int_{\partial\Delta}(\lambda-A_2)^{-1}\pi_2E(\lambda)^{-1}\tau_1 d\lambda, \tag{5.2}$$

its inverse is equal to

$$S^{-1} = (2\pi i)^{-1}\int_{\partial\Delta}\pi_1F(\lambda)^{-1}\tau_2(\lambda-A_2)^{-1}d\lambda. \tag{5.3}$$

Here Δ is some bounded Cauchy domain such that $(\sigma(A_1)\cup\sigma(A_2))\subset\Delta\subset\overline{\Delta}\subset\Omega$, the map $\pi_i: X_i\oplus Z_i\to X_i$ is the projection of $X_i\oplus Z_i$ onto X_i and $\tau_i: X_i\to X_i\oplus Z_i$ is the natural embedding of X_i into $X_i\oplus Z_i$.

Proof. Obviously, A_1 and A_2 are spectral linearizations of $W(\lambda) = \lambda - A_1$ on Ω. So the similarity of A_1 and A_2 is clear from Theorem 5.1.

Using the construction contained in the proof of Theorem 5.1 we get a spectral node $\theta_2 = (A_2,B,C;X_2,X_1)$ for $W(\lambda) = \lambda - A_1$ on Ω. Another spectral node for W on Ω is $\theta_1 = (A_1,I,I;X_1,X_1)$. We now apply Theorem 1.2 to the spectral nodes θ_1 and θ_2, and obtain Eqs. (5.2) and (5.3). □

In the finite-dimensional case the previous theorem is trivial, because in that case the equivalence of $(\lambda-A_1)\oplus I_{Z_1}$ and $(\lambda-A_2)\oplus I_{Z_2}$ implies that A_1 and A_2 have the same Jordan form. Using spectral factorization it has been proved in [25] that (infinite-dimensional) operators A_1 and A_2 are similar if and only if $\lambda - A_1$ and $\lambda - A_2$ are equivalent on an open disk containing $\sigma(A_1)\cup\sigma(A_2)$, which is a particular case of Theorem 5.2.

We consider now equivalence of analytic operator functions. The next theorem, which may be viewed as a more general version of the first part of Theorem 5.2, answers (in terms of spectral linearizations) the question when after extension two analytic operator functions are equivalent.

THEOREM 5.3. *For* $i = 1,2$ *let* $W_i\colon \Omega \to L(Y)$ *be an analytic operator function with compact spectrum, and let* $A_i\colon X_i \to X_i$ *be a spectral linearization of* W_i *on* Ω. *Then the following statements are equivalent:*

(i) *there exist Banach spaces* Z_1 *and* Z_2 *such that* $W_1(\lambda) \oplus I_{Z_1}$ *and* $W_2(\lambda) \oplus I_{Z_2}$ *are equivalent on* Ω;

(ii) $W_1(\lambda) \oplus I_{X_1}$ *and* $W_2(\lambda) \oplus I_{X_2}$ *are equivalent on* Ω;

(iii) A_1 *and* A_2 *are similar.*

Proof. From Theorem 5.1 we know that for $i = 1,2$ the operator A_i is the main operator of a spectral node θ_i for W_i on Ω. So we may apply Corollary 4.2 and conclude that for $i = 1,2$ the operator functions $W_i(\lambda) \oplus I_{X_i}$ and $(\lambda - A_i) \oplus I_Y$ are equivalent on Ω.

If (i) holds true, then the operator functions $(\lambda - A_1) \oplus I_Y \oplus I_{X_2} \oplus I_{Z_1}$ and $(\lambda - A_2) \oplus I_Y \oplus I_{X_1} \oplus I_{Z_2}$ are equivalent on Ω. By Theorem 5.2, (iii) follows.

Next, suppose (iii) is true. Then the operator functions $(\lambda - A_1) \oplus I_Y$ and $(\lambda - A_2) \oplus I_Y$ are equivalent on Ω, and (ii) follows.

Obviously, (ii) implies (i). □

In Theorem 5.3 we assume that the values of the operator functions W_1 and W_2 act on one and the same space Y. If we drop this condition, then with appropriate modification the theorem remains true. In fact, if $W_i\colon \Omega \to L(Y_i)$, then Theorem 5.3 holds provided statement (ii) is replaced by

(ii)' $W_1(\lambda) \oplus I_{X_1} \oplus I_{Y_2}$ and $W_2(\lambda) \oplus I_{X_2} \oplus I_{Y_1}$ are equivalent on Ω.

The next example (taken from [10], Example 1.2) shows that in general similarity of spectral linearizations does not imply equivalence of the original operator functions.

EXAMPLE 5.4. Let $Y = \ell_2$ and Ω the unit disk. Then we define analytic operator functions $W_1, W_2 \colon \Omega \to L(\ell_2)$ by

$$W_1(\lambda)(x_0, x_1, x_2, \ldots) = (\ x_0, (1-2\lambda)x_1, (1-2\lambda)x_2, \ldots);$$

$$W_2(\lambda)(x_0, x_1, x_2, \ldots) = ((1-2\lambda)x_0, (1-2\lambda)x_1, (1-2\lambda)x_2, \ldots).$$

Then $\Sigma(W_1) = \Sigma(W_2) = \{\frac{1}{2}\}$ and a spectral linearization of W_1 and W_2 on Ω is given by

$$A(x_0, x_1, x_2, \ldots) = (\tfrac{1}{2}x_0, \tfrac{1}{2}x_1, \tfrac{1}{2}x_2, \ldots) .$$

However, since $W_1(\frac{1}{2}) \neq 0$ and $W_2(\frac{1}{2}) = 0$, the operator functions W_1 and W_2 cannot be equivalent on Ω.

As we have noticed in the introduction, the companion operator C_L of a monic operator polynomial $L(\lambda)$ is a spectral linearization of L on $\mathbb{C}$, and hence, by Theorem 5.1, is the main operator of a spectral node for $L(\lambda)$ on any open set containing $\Sigma(L)$. Since the main operator of a spectral node does not change if the operator function is replaced by an equivalent operator function (cf. Theorem 2.3), we obtain the following corollary (cf. [25], Theorem 1, for a weaker version):

COROLLARY 5.5. *Two monic operator polynomials $L_1(\lambda)$ and $L_2(\lambda)$ with coefficients acting on Y have similar companion operators if and only if there exist Banach spaces Z_1 and Z_2 such that $L_1(\lambda) \oplus I_{Z_1}$ and $L_2(\lambda) \oplus I_{Z_2}$ are equivalent on $\mathbb{C}$.*

Next we make a few remarks about Theorem 5.3. Let $W_1, W_2\colon \Omega \to L(Y)$ be analytic operator functions with compact spectrum. If W_1 and W_2 have the same spectral linearization, then according to Theorem 5.3, the functions $W_1(\lambda) \oplus I_X$ and $W_2(\lambda) \oplus I_X$ are equivalent on Ω, where X is the space on which the spectral linearization acts. However, in certain cases $W_1(\lambda) \oplus I_Z$ and $W_2(\lambda) \oplus I_Z$ are equivalent on Ω for "smaller" spaces Z. For instance, if $\dim Y < \infty$, then $W_1(\lambda)$ and $W_2(\lambda)$ are equivalent (this follows from the fact that analytic matrix functions are equivalent if and only if they have the same spectra and equal partial multiplicities at each spectral point (see [17], Theorem 3.3)). Another example is the case when W_1 and W_2 are monic operator polynomials of degree ℓ. Indeed, if W_1 and W_2 have similar companion operators (which act on Y^ℓ), then $W_1(\lambda) \oplus I_{Y^{\ell-1}}$ is equivalent to $W_1(\lambda) \oplus I_{Y^{\ell-1}}$ on $\mathbb{C}$ (see [12], Section 2.1), while the spaces $Y^{\ell-1}$ and Y^ℓ are not necessarily isomorphic (even if Y is infinite dimensional; see, for instance, [8]).

Returning to Example 5.4 note that in this example $W_1(\lambda) \oplus I_{\mathbb{C}^k}$ is not equivalent to $W_2(\lambda) \oplus I_{\mathbb{C}^k}$ for any positive integer k. As the next theorem shows, this is always the case when W_1 and W_2 are not equivalent on Ω.

THEOREM 5.6. *Let $W_i\colon \Omega \to L(Y)$, $i = 1,2$ be analytic operator functions (not necessarily with compact spectra). Assume that $W_1(\lambda) \oplus I_{\mathbb{C}^k}$ and $W_2(\lambda) \oplus I_{\mathbb{C}^k}$ are equivalent on Ω. Then $W_1(\lambda)$ and $W_2(\lambda)$ are equivalent on Ω as well.*

Proof. Apparently, it suffices to assume $k = 1$; omitting the variable λ,

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & e \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} W_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & f \end{bmatrix} \tag{5.4}$$

where $E_{11}, F_{11}\colon \Omega \to L(Y)$; $E_{12}, F_{12}\colon \Omega \to L(\mathbb{C}, Y)$; $E_{21}, F_{21}\colon \Omega \to L(Y, \mathbb{C})$; $e, f\colon \Omega \to L(\mathbb{C})$ are analytic, and the values of

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & e \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & f \end{bmatrix}$$

are invertible operators for every $\lambda \in \Omega$ (e and f will be considered as scalar analytic functions; in fact, $e = f$).

First consider the case when $e(\lambda) \neq 0$ for all $\lambda \in \Omega$. Multiplying (5.4) from the left and from the right by

$$\begin{bmatrix} I & -E_{12}e^{-1} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 \\ -f^{-1}F_{21} & 1 \end{bmatrix},$$

respectively, we obtain:

$$\begin{bmatrix} E_{11}-E_{12}e^{-1}E_{21} & 0 \\ E_{21} & e \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ -f^{-1}F_{21} & 1 \end{bmatrix} = \begin{bmatrix} W_2 & E_{12}e^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{11}-F_{12}f^{-1}F_{21} & F_{12} \\ 0 & f \end{bmatrix}.$$

In particular, $(E_{11}-E_{12}e^{-1}E_{21})W_1 = W_2(F_{11}-F_{12}f^{-1}F_{21})$. As

$E_{11}-E_{12}e^{-1}E_{21}$ and $F_{11}-F_{12}f^{-1}F_{21}$ are invertible for all $\lambda \in \Omega$, the equivalence of W_1 and W_2 follows.

Consider now the general case. We can assume $E_{12}(\lambda) \not\equiv 0$ (otherwise the invertibility of

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & e \end{bmatrix}$$

for all $\lambda \in \Omega$ would imply that $e(\lambda) \neq 0$ for all $\lambda \in \Omega$, the case considered above). Regard $E_{12}(\lambda)$ as an analytic Y-valued function ,and let $\lambda_1,\lambda_2,\ldots$ be all the zeros of $E_{12}(\lambda)$ in Ω with multiplicities $\mu_1,\mu_2,\ldots$, respectively (if $E_{12}(\lambda)$ has no zeros, the changes in the subsequent reasoning are obvious). Let $\varphi(\lambda)$ be a scalar analytic (in Ω) function with zeros $\lambda_1,\lambda_2,\ldots$ and corresponding multiplicities $\mu_1,\mu_2,\ldots$. Then $\varphi(\lambda)^{-1}E_{12}(\lambda)$ is analytic Y-valued function with no zeros. By Allan's theorem [1] there exists an analytic function $X\colon \Omega \to L(Y,\mathbb{C})$ such that $X(\lambda)(\varphi(\lambda)^{-1}E_{12}(\lambda)) \equiv 1$; in other words,

$$XE_{12} = \varphi \ . \tag{5.5}$$

Observe that the set of zeros of E_{12} (which is exactly the set of zeros of φ) and the set of zeros of e are disjoint; as proved in [6] (see also S. Friedland, Spectral theory of matrices. I. General matrices, Research report, Mathematics Research Center, University of Wisconsin, Madison, 1980), there exists an analytic (in Ω) scalar function ψ such that the function $\psi\varphi + e$ has no zeros in Ω. Now multiply (5.4) from the left by

$$\begin{bmatrix} I & 0 \\ \psi X & 1 \end{bmatrix} .$$

In view of (5.5) we have:

$$\begin{bmatrix} E_{11} & E_{12} \\ \psi X E_{11}+E_{21} & \psi\varphi+e \end{bmatrix}\begin{bmatrix} W_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} W_2 & 0 \\ \psi X W_2 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ -\psi X W_2 & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I & 0 \\ \psi X W_2 & 1 \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & f \end{bmatrix} =$$

$$= \begin{bmatrix} W_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ \psi X W_2 F_{11} + F_{21} & \psi X W_2 F_{12} + f \end{bmatrix} .$$

Using this equality in place of (5.4), we reduce the proof to the case when $e(\lambda) \neq 0$ for all $\lambda \in \Omega$, which has been considered already. □

Results analogous to Theorem 5.6 have been proved in [6] in different contexts; for example, for analytic matrix valued functions with finite number of essential singularities. For operator functions Theorem 5.6 seems to be new.

If in Example 5.4 we allow different extension spaces, then one can make equivalence with finite-dimensional extensions. In fact, $W_1(\lambda)$ is equivalent to $W_2(\lambda) \oplus I_{\mathbb{C}}$ in this example. Hence it is natural to consider the following notion of minimal space extension.

Let $W_i\colon \Omega \to L(Y)$, $i = 1,2$ be analytic (with compact spectrum), and suppose that $W_1(\lambda) \oplus I_{Z_1}$ is equivalent to $W_2(\lambda) \oplus I_{Z_2}$ on Ω, for some Banach spaces Z_1 and Z_2 (by Theorem 5.3 this means similarity of the spectral linearizations of W_1 and W_2). The pair (Z_1, Z_2) will be called a *minimal space extension* for W_1 and W_2, if the following property holds: If Z_1' and Z_2' are Banach spaces such that Z_i' is isomorphic to a subspace in Z_i, $i = 1,2$, and $W_1(\lambda) \oplus I_{Z_1'}$ and $W_2(\lambda) \oplus I_{Z_2'}$ are equivalent on Ω, then Z_i' is isomorphic to Z_i, $i = 1,2$. Above we have seen that $(0,0)$ is always the minimal space extension in the case when $\dim Y < \infty$. In Example 5.4 the pair $(0, \mathbb{C})$ is obviously a minimal space extension.

In case when Y is the infinite dimensional separable Hilbert space, a minimal space extension for given W_1 and W_2 exists always. In the following theorem we list all the possibilities that may occur in this case.

THEOREM 5.7. *Let Y be the infinite dimensional separable Hilbert space, and let $W_i\colon \Omega \to L(Y)$, $i = 1,2$, be analytic operator functions with compact spectra. Assume that the spectral linearizations of W_1 and W_2 are similar. Then one of the following pairs of spaces is a minimal space extension for W_1 and*

W_2: (Y,Y); $(0,Y)$; $(Y,0)$; $(0,\mathbb{C}^k)$, $k = 1,2,\ldots$; $(\mathbb{C}^k,0)$, $k = 1,2,\ldots$; $(0,0)$.

<u>Proof</u>. Let $A_i: X \to X$ be a spectral linearization for $W_i(\lambda)$, $i = 1,2$. As Y is a separable Hilbert space, we may assume X is a separable Hilbert space too, and therefore, since $\dim Y = \infty$, we may assume that X is a subspace of Y. Further, by Theorem 5.3, $W_1(\lambda) \oplus I_X$ and $W_2(\lambda) \oplus I_X$ are equivalent. Now the theorem follows easily from Theorem 5.6. □

Note that all possibilities in Theorem 5.7 can be realized. Indeed, put $W_2(\lambda) = (1-2\lambda)I$, and

$$W_1(\lambda) = \begin{bmatrix} I_{Z_1} & 0 \\ 0 & (1-2\lambda)I_{Z_2} \end{bmatrix},$$

where $Y = Z_1 \oplus Z_2$ with $\dim Z_2 = \infty$ (cf. Example 5.4). Then $(0,Y)$ or $(0,\mathbb{C}^k)$ is a minimal space extension for W_1 and W_2 according if $\dim Z_1 = \infty$ or $\dim Z_1 = k$. Interchanging the roles of W_1 and W_2 we obtain examples of minimal space extensions $(Y,0)$ and $(\mathbb{C}^k,0)$. The following is an example where (Y,Y) is a minimal space extension.

EXAMPLE 5.8. Let $\lambda_1 \neq \lambda_2$ be complex numbers, and put

$$W_1(\lambda) = \begin{bmatrix} (\lambda-\lambda_1)(\lambda-\lambda_2)I_Y & 0 \\ 0 & (\lambda-\lambda_1)I_Y \end{bmatrix};$$

$$W_2(\lambda) = \begin{bmatrix} (\lambda-\lambda_1)(\lambda-\lambda_2)I_Y & 0 \\ 0 & (\lambda-\lambda_2)I_Y \end{bmatrix}.$$

It is easily seen that $W_1(\lambda) \oplus I_Y$ and $W_2(\lambda) \oplus I_Y$ are equivalent on $\mathbb{C}$ (this follows form the equality

$$\begin{bmatrix} \alpha & -\alpha(\lambda-\lambda_2)+1 \\ 1 & -(\lambda-\lambda_2) \end{bmatrix}\begin{bmatrix} (\lambda-\lambda_1)(\lambda-\lambda_2) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ \lambda-\lambda_1 & -\alpha(\lambda-\lambda_1)-1 \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda-\lambda_1 & 0 \\ 0 & \lambda-\lambda_2 \end{bmatrix},$$

where $\alpha = (\lambda_1-\lambda_2)^{-1}$). On the other hand, if $W_1(\lambda) \oplus I_{Z_1}$ and

$W_2(\lambda) \oplus I_{Z_2}$ are equivalent, then

$$\dim Z_1 = \dim \operatorname{Im} [W_1(\lambda_1) \oplus I_{Z_1}] = \dim \operatorname{Im} [W_2(\lambda_1) \oplus I_{Z_2}] = \infty ,$$

and analogously $\dim Z_2 = \infty$. So indeed (Y,Y) is a minimal space extension for W_1 and W_2.

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