The number of bound states of the Coulomb plus Yamaguchi potential

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It is shown that certain assertions on the number of bound states of a Coulomb plus Yamaguchi potential which Zachary [J. Math. Phys. **12**, 1379 (1971); **14**, 2018 (1973)] claims to have proved are incorrect. We prove that there are always infinitely many bound states if the Coulomb part of the potential is attractive and that, in case the Coulomb part of the potential is repulsive, there is one bound state only if the Yamaguchi potential is sufficiently attractive.

In this paper we correct some assertions which Zachary[1] claims to have proved concerning the number of bound states (in the $l=0$ partial wave projected space) for the Coulomb plus Yamaguchi potential.

We prove that the number of s wave bound states is always infinite if the Coulomb part of the potential is attractive, for a repulsive as well as for an attractive Yamaguchi potential. Zachary found (by means of numerical calculations) that the number of bound states would be 0 or 1 in this case.

In case the Coulomb part of the potential is repulsive, we prove that there is one and only one bound state if the Yamaguchi potential is sufficiently attractive, and that there is no bound state otherwise. Zachary found in this case that the number of bound states could be 0, 1, or 2. See Ref. 1, pp. 1384 and 1385.

We start with the observation that all the bound states are given by the poles of the T operator. In the notation of Ref. 2, we have $V=V_c+V_s$. Here V_s is the rank-one separable Yamaguchi potential with strength λ and range parameter β. V_s is attractive or repulsive when $\lambda>0$ or $\lambda<0$, respectively. Further, V_c is the pure Coulomb potential with strength s, V_c being attractive when $s>0$ and repulsive when $s<0$. Furthermore we shall use the variable κ which is connected to the energy by $E=-\kappa^2$, $\kappa>0$. Then

$$T=T_c+T_{cs}, \tag{1}$$

$$T_{cs}=-\frac{|g^c\rangle\langle g^c|}{\lambda^{-1}+\langle g|G_c|g\rangle}, \tag{2}$$

where g^c is the Coulomb-modified form factor. When V_c is repulsive, neither T_c nor g^c has poles. Below we shall show[3] that, when V_c is attractive, the pure Coulomb poles in T_c are cancelled by corresponding poles in T_{cs}. Then it follows that the poles of T are obtained by solving the equation

$$\lambda^{-1}+\langle g|G_c|g\rangle=0. \tag{3}$$

In the case g is the Yamaguchi form factor, the second term is known in closed form [cf. Eq. (83) of Ref. 2] and we have

$$\lambda^{-1}=\frac{1}{2\beta(\beta+\kappa)^2}\,\frac{1}{1-s/\kappa}\;{}_2F_1(1,-s/\kappa;2-s/\kappa;[(\beta-\kappa)/(\beta+\kappa)]^2). \tag{4}$$

This is essentially Eq. (31) of Zachary.

We now first consider the case that V_c is attractive, i.e., $s>0$. In that case T_c has the pure Coulomb bound-state poles at $\kappa=s/n$, $n=1, 2,\cdots$. The origin $\kappa=0$ is the limit point of these poles. However, we do not expect bound states of V_c+V_s at *these* energies. In fact, T_{cs} has poles at exactly the same points $\kappa=s/n$ and its residues cancel the residues of T_c. It follows that T_c+T_{cs} has for these values of κ "removable poles" (in the terminology of Ref. 4). This can be shown in the following way. In the neighborhood of the point $\kappa=s/n$, where we fix n for the moment, we have

$$T_c\approx\frac{G_0^{-1}|\kappa_n\rangle\langle\kappa_n|G_0^{-1}}{-\kappa^2+s^2/n^2}\quad(\kappa\approx s/n), \tag{5}$$

where $|\kappa_n\rangle$ is the pure Coulomb bound state vector. Using then

$$G_c=G_0+G_0T_cG_0\approx G_0T_cG_0,$$
$$|g^c\rangle=(1+T_cG_0)|g\rangle\approx T_cG_0|g\rangle,$$

where both approximations hold near the pure Coulomb bound state poles, we get from Eqs. (1) and (2),

$$T\approx T_c-\frac{T_cG_0|g\rangle\langle g|G_0T_c}{\langle g|G_0T_cG_0|g\rangle}. \tag{6}$$

Insertion of Eq. (5) into Eq. (6) shows that the residues of T_c and T_{cs} cancel,[3] i.e.,

$$\lim_{\kappa\to s/n}(-\kappa^2+s^2/n^2)T=0. \tag{7}$$

It is also clarifying to consider the following interesting equality, which holds without approximation,

$$\langle g|G|g\rangle^{-1}=\lambda+\langle g|G_c|g\rangle^{-1}. \tag{8}$$

Clearly, the poles of $\langle g|G_c|g\rangle$ are no poles of $\langle g|G|g\rangle$ (and vice versa) as long as $\lambda\neq0$. Furthermore, the resolvent G and the T operator have the same poles, which follows easily from

$$G=G_0+G_0TG_0.$$

We now turn to the solution of Eq. (4). All the variables in Eq. (4) are real and it follows that the whole expression is real. Due to $s>0$ we have $-s/\kappa<0$. Now it is known that ${}_2F_1(a,b;c;z)/\Gamma(c)$ is an entire analytic function of a, b, and c if z is fixed and $|z|<1$. It follows that the expression on the right-hand side of Eq. (4) has simple poles at $s/\kappa=n=1, 2, \cdots$. (These are just the pure Coulomb bound state poles.) At such a pole it behaves as[3]

$$(n-s/\kappa)^{-1}2s(\beta-s/n)^{2n-2}(\beta+s/n)^{-2n-2},$$

from which it follows that the residues have the same sign for all n. Therefore, if we vary κ from $s/(n+1)$ to s/n (i.e., between any pair of consecutive poles), that expression varies continuously from $+\infty$ to $-\infty$ and adopts every real number at least once. (Below we shall find that it adopts every real number *just* once.) This holds for every $n=1, 2, \cdots$, so Eq. (4) has infinitely many solutions for every real value of λ, i.e., there is a bound state corresponding to $\kappa=s/n$ ($n=1, 2, \cdots$) for an arbitrarily strongly repulsive or attractive Yamaguchi potential. The origin $\kappa=0$ (zero energy) is the only accumulation point of the bound state energies.

A second way to prove this, which at the same time gives more detailed information about the position of the bound state energies with respect to the pure Coulomb bound states, is to insert the completeness relation

$$\mathbb{1}=\sum_{n=1}|\kappa_n\rangle\langle\kappa_n| + \int_0^\infty dk\,k^2|k+\rangle\langle k+| \tag{9}$$

into Eq. (3). Here again $|\kappa_n\rangle$ are the bound state vectors and $|k+\rangle$ are the scattering states of the attractive pure Coulomb potential. Using then $G_c=-(\kappa^2+H_c)^{-1}$ where $H_c=H_0+V_c$, we get

$$\lambda^{-1}=\sum_{n=1}^{\infty}\frac{\langle g|\kappa_n\rangle\langle\kappa_n|g\rangle}{\kappa^2-s^2/n^2} + \int_0^\infty\frac{dk\,k^2}{\kappa^2+k^2}\langle g|k+\rangle\langle k+|g\rangle. \tag{10}$$

The integrand and each term of the infinite sum is a monotonically decreasing function of κ on each of the intervals $s/(n+1)<\kappa<s/n$, $n=0, 1,\cdots$; This can be seen either by inspection or by means of differentiation with respect to κ^2. It follows that the right-hand side of Eq. (10) is a monotonically decreasing function of κ on the above intervals. So if κ increases between any pair of adjacent poles[3] [from $s/(n+1)$ to s/n, say], the expression on the right-hand side of Eq. (10) *decreases continuously and monotonically* from $+\infty$ to $-\infty$. Therefore, Eq. (10) has for every real value of λ one and only one solution in the interval $s/(n+1)<\kappa<s/n$, for $n=1, 2,\cdots$. Furthermore, in the $n=0$ interval $s<\kappa<\infty$ there is one and only one solution for every real *positive* value of λ, since the right-hand side of Eq. (10) varies then continuously and monotonically from $+\infty$ to 0.

This means that in the case of an attractive Yamaguchi part there is just one bound state below the pure Coulomb ground state, with binding energy $E_B<-s^2$. This is the ground state of V_c+V_s. By increasing the Yamaguchi strength, $\lambda\to\infty$, we get an infinite binding energy as expected, $E_B\to\infty$. Also all other bound states of V_c+V_s, namely those with $n=2,3,\cdots$, are shifted downwards with respect to the corresponding pure Coulomb bound states. But in this case the bound states always remain above the next lower pure Coulomb bound states. On the other hand, if the Yamaguchi part is repulsive, all bound states are shifted upwards with respect to the pure Coulomb bound states, but every state remains below the next higher pure Coulomb state, no matter how strongly repulsive V_s is. This is a remarkable and quite unexpected phenomenon.

Now we consider the case that V_c is repulsive. In this case the pure Coulomb scattering states $|k+\rangle$ form a complete set in the $l=0$ space. The completeness relation now takes the form

$$1=\int_0^\infty dk\,k^2|k+\rangle\langle k+|. \tag{11}$$

Again, we insert Eq. (11) into Eq. (3) and use the fact that $G_c=-(\kappa^2+H_c)^{-1}$ with $H_c|k+\rangle=k^2|k+\rangle$. Then Eq. (3) becomes

$$\lambda^{-1}-\int_0^\infty\frac{dk\,k^2}{\kappa^2+k^2}\langle g|k+\rangle\langle k+|g\rangle=0. \tag{12}$$

The integrand is clearly real positive and it is a continuous and monotonically decreasing function of κ for $0<\kappa<\infty$. The same holds for the integral. It is maximal for $\kappa=0$. We denote the corresponding strength by λ_0,

$$\lambda_0^{-1}=\int_0^\infty dk\langle g|k+\rangle\langle k+|g\rangle. \tag{13}$$

It follows by inspection that Eq. (12) has one and only one solution if the Yamaguchi potential is sufficiently attractive, i.e., if $\lambda\geq\lambda_0$. When $\lambda<\lambda_0$ there is no solution and therefore no bound state. An explicit expression for λ_0 follows from

$$\beta^3/\lambda_0=\tfrac{1}{2}-2\nu\exp(4\nu)\,\Gamma(0,4\nu), \tag{14}$$

in the notation of Ref. 5 ($\nu=-s/\beta>0$). For this value of λ_0, the Coulomb-modified scattering length is infinite, $a_{cs}^{-1}=0$, see Eq. (34) of Ref. 5. We notice that

$$0<x\exp(x)\Gamma(0,x) = x\exp(x)\int_x^\infty dt\,t^{-1}\exp(-t)<1,\quad x>0,$$

so that the right-hand side of Eq. (14) is always positive and therefore $\lambda_0^{-1}>0$, cf. Eq. (13). It also follows from Eq. (14) that $\beta^3/\lambda_0<\frac{1}{2}$. This is satisfactory since it is known that for the *pure* Yamaguchi potential the bound state appears just at zero energy if the strength λ is equal to $2\beta^3$. Addition of the repulsive Coulomb potential must have the effect that $\lambda_0>2\beta^3$.

Finally we note that Eq. (12) implies that the expression on the right-hand side of Eq. (4) is a continuous and monotonically decreasing function of κ for $0<\kappa<\infty$ if $s<0$. This can be proved directly, but with considerably more effort, as follows. Starting from a well-known integral representation for the hypergeometric function, we can recast Eq. (4) into the form

$$\lambda^{-1}=2\kappa\int_0^1 dt\,t^{-s/\kappa}N^{-2}, \tag{15}$$

with

$$N=(\beta+\kappa)^2-t(\beta-\kappa)^2\geq 0.$$

Here we have also utilized

$${}_2F_1(1,i\gamma;2+i\gamma;z)=(1-z)\,{}_2F_1(2,1+i\gamma;2+i\gamma;z).$$

We differentiate the right-hand side of Eq. (15) with respect to κ and obtain, after a few partial integrations,

$$2\int_0^1 dt\,t^{-s/\kappa}N^{-3}\{2(\beta^2-\kappa^2)(1-t)+\ln t[(\beta+\kappa)^2+t(\beta-\kappa)^2]\}. \tag{16}$$

One easily verifies that

$$\ln t=\sum_{n=1}^{\infty}\frac{2}{2n-1}\left(\frac{t-1}{t+1}\right)^{2n-1}<2\frac{t-1}{t+1},\quad 0<t<1.$$

Substitution of this inequality shows that the integrand is dominated by

$$t^{-s/\kappa}N^{-3}\,4\kappa(t-1)[\kappa+\beta(1-t)/(1+t)],$$

which is clearly negative for $0<t<1$, so that the integral of Eq. (16) is also negative. This proves the monotonicity.

We note that almost all the assertions of this paper remain valid when the Yamaguchi potential is replaced by an arbitrary rank-one separable potential. It is not difficult to verify this. There is one important exception, however. When we discussed the solutions of Eq. (4), we assumed the energy to be negative. It can be shown, with the help of Eqs. (10) and (12), that Eq. (4) has indeed no solution for positive energy if g is the Yamaguchi form factor. That is, there is no bound state in the continuum. However, by a special choice of the form factor it is possible to construct a bound state at positive energy. Such a pathological situation will not be discussed here.

The results of this paper agree with our intuitive idea, namely that the range of V_c+V_s, being still infinite, causes an infinite number of bound states in case V_c is attractive. On the other hand, it is known that an attractive rank-one separable potential has at most one bound state at negative energy. Addition of a repulsive Coulomb potential should not change the situation.

The mistake of Zachary shows that the hypergeometric functions occurring here are complicated objects. The source of the difficulties is that the energy variable is contained in the parameters of ${}_2F_1$ as well as in its argument. In particular in the zero energy region one should be careful. Numerical calculations might fail here because the well known ordinary power series of ${}_2F_1$ converges very slowly. A method for practical calculations, in particular useful in this region, has been developed in Ref. 4.

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[1]W. W. Zachary, J. Math. Phys. **12**, 1379 (1971); **14**, 2018 (1973).

[2]H. van Haeringen and R. van Wageningen, J. Math. Phys. **16**, 1441 (1975).

[3]We assume that $\langle g|\kappa_n\rangle \neq 0$. When g is the Yamaguchi form factor this means that $s \neq n\beta$ for $n=2,3,\cdots$.

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