Solving Finite Block Toeplitz Systems and Systems of Finite-section Convolution Equations

1 Solving Finite Block Toeplitz Systems

sec:1

1. Auxiliary Linear Systems. Let $(H_i)_{-(p-1)}^{p-1}$ be a sequence of complex $k \times k$ matrices. Consider the finite block Toeplitz matrix

$$A^{(p)} = \left(H_{i-j}\right)_{i,j=1}^{p}.$$

For convenience we also define the finite block Toeplitz matrix

$$\tilde{A}^{(p)} = \left(H_{j-i}\right)_{i,j=1}^{p}, \qquad \tilde{A}^{(p)} = M^{(p)} A^{(p)} M^{(p)},$$

where $M^{(p)} = \left[\delta_{i+j,p+1} I_k\right]_{i,j=1}^{p}$. Consider also the following linear systems

$$\sum_{j=1}^{p} H_{i-j} X_{j-1} = \delta_{i,1} I_k, \qquad i = 1, \cdots, p; \tag{1.1}$$ eq:1.1

$$\sum_{j=1}^{p} H_{j-i} Z_{-(j-1)} = \delta_{i,1} I_k, \qquad i = 1, \cdots, p; \tag{1.2}$$ eq:1.2

$$\sum_{j=1}^{p} W_{j-1} H_{i-j} = \delta_{i,1} I_k, \qquad i = 1, \cdots, p; \tag{1.3}$$ eq:1.3

$$\sum_{j=1}^{p} Y_{-(j-1)} H_{j-i} = \delta_{i,1} I_k, \qquad i = 1, \cdots, p, \tag{1.4}$$ eq:1.4

where $\delta_{i,j}$ is the Kronecker delta and I_k denotes the identity matrix of order k. Then these four systems can be written in the concise form

$$A^{(p)} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{p-1} \end{bmatrix} = \begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \tilde{A}^{(p)} \begin{bmatrix} X_{p-1} \\ X_{p-2} \\ \vdots \\ X_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}; \tag{1.5}$$ eq:1.5

$$\tilde{A}^{(p)} \begin{bmatrix} Z_0 \\ Z_{-1} \\ \vdots \\ Z_{-(p-1)} \end{bmatrix} = \begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad A^{(p)} \begin{bmatrix} Z_{-(p-1)} \\ Z_{-(p-2)} \\ \vdots \\ Z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}; \tag{1.6}$$ eq:1.6

$$\begin{cases} \begin{bmatrix} W_0 & \cdots & W_{p-1} \end{bmatrix} \tilde{A}^{(p)} = \begin{bmatrix} I_k & 0 & \cdots & 0 \end{bmatrix}, \\ [2mm] \begin{bmatrix} W_{p-1} & \cdots & W_0 \end{bmatrix} A^{(p)} = \begin{bmatrix} 0 & \cdots & 0 & I_k \end{bmatrix}; \end{cases} \tag{1.7}$$ eq:1.7

$$\begin{cases} \begin{bmatrix} Y_0 & \cdots & Y_{-(p-1)} \end{bmatrix} A^{(p)} = \begin{bmatrix} I_k & 0 & \cdots & 0 \end{bmatrix}, \\ [2mm] \begin{bmatrix} Y_{-(p-1)} & \cdots & Y_0 \end{bmatrix} \tilde{A}^{(p)} = \begin{bmatrix} 0 & \cdots & 0 & I_k \end{bmatrix}. \end{cases} \tag{1.8}$$ eq:1.8

Then $X_0, \cdots, X_{p-1}$ occupy the first column, $Z_{-(p-1)}, \cdots, Z_0$ the last column, $Y_0, \cdots, Y_{-(p-1)}$ the first row, and $W_{p-1}, \cdots, W_0$ the last row of the inverse of $A^{(p)}$ (if it exists). As a result, we have

$$X_0 = Y_0, \quad Z_0 = W_0, \quad X_{p-1} = W_{p-1}, \quad Y_{-(p-1)} = Z_{-(p-1)},$$

provided the inverse of $A^{(p)}$ exists. In fact, the solvability of Eqs. (1.1)-(1.4) [or Eqs. (1.5)-(1.8)] is equivalent to the nonsingularity of the matrix $A^{(p)}$. In that case the solutions of these equations give us the "edge" of the matrix $[A^{(p)}]^{-1}$.

2. Recursive Inversion. Now consider the four matrices

$$X = \begin{bmatrix} X_0 & 0 & \cdots & \cdots & 0 \\ X_1 & I_k & 0 & \cdots & 0 \\ \vdots & 0 & I_k & & \vdots \\ \vdots & \vdots & & \ddots & \\ X_{p-1} & 0 & \cdots & 0 & I_k \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_0 & Y_1 & \cdots & \cdots & Y_{-(p-1)} \\ 0 & I_k & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I_k \end{bmatrix};$$

$$Z = \begin{bmatrix} I_k & 0 & \cdots & 0 & Z_{-(p-1)} \\ 0 & I_k & 0 & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & I_k & Z_{-1} \\ 0 & \cdots & \cdots & 0 & Z_0 \end{bmatrix}, \quad W = \begin{bmatrix} I_k & 0 & \cdots & \cdots & 0 \\ 0 & I_k & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & I_k & 0 \\ W_{p-1} & \cdots & \cdots & W_1 & W_0 \end{bmatrix},$$

where only the first block column of X, the first block row of Y, the last block column of Z, and the last block row of W differ from the corresponding block row/column of the identity matrix of order kp. Then one easily verifies that $A^{(p)}Z$ is obtained from $A^{(p)}$ by replacing its last block column by the last block column of the identity matrix of order kp and that $A^{(p)}X$ is obtained from $A^{(p)}$ by replacing its first block column by the first block column of the identity matrix of order kp. As a result,

$$WA^{(p)}Z = A^{(p-1)} \oplus W_0, \qquad YA^{(p)}X = Y_0 \oplus A^{(p-1)}. \tag{1.9}$$

eq:1.9

Consequently,

$$\begin{aligned} [A^{(p)}]^{-1} &= Z\left([A^{(p-1)}]^{-1} \oplus W_0^{-1}\right) W \\ &= X\left(Y_0^{-1} \oplus [A^{(p-1)}]^{-1}\right) Y. \end{aligned} \tag{1.10}$$

eq:1.10

Writing $C^{(p)} = [A^{(p)}]^{-1}$ and introducing the column and row matrices

$$\boldsymbol{Z} = \begin{bmatrix} Z_{-(p-1)} \\ \vdots \\ Z_{-1} \end{bmatrix}, \qquad \boldsymbol{W} = \begin{bmatrix} W_{p-1} & \cdots & W_1 \end{bmatrix};$$

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_{p-1} \end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix} Y_{-1} & \cdots & Y_{-(p-1)} \end{bmatrix},$$

we easily obtain, using $Z_0 = W_0$ and $X_0 = Y_0$,

$$\begin{aligned} C^{(p)} &= Z\left([A^{(p-1)}]^{-1} \oplus W_0^{-1}\right) W \\ &= \begin{bmatrix} I_{k(p-1)} & \boldsymbol{Z} \\ 0 & Z_0 \end{bmatrix} \begin{bmatrix} C^{(p-1)} & 0 \\ 0 & W_0^{-1} \end{bmatrix} \begin{bmatrix} I_{k(p-1)} & 0 \\ \boldsymbol{W} & W_0 \end{bmatrix} \\ &= \begin{bmatrix} C^{(p-1)} + \boldsymbol{Z}W_0^{-1}\boldsymbol{W} & \boldsymbol{Z} \\ \boldsymbol{W} & W_0 \end{bmatrix} \end{aligned} \tag{1.11}$$

eq:1.11

and

$$\begin{aligned} C^{(p)} &= X\left(Y_0^{-1} \oplus [A^{(p-1)}]^{-1}\right) Y \\ &= \begin{bmatrix} X_0 & 0 \\ \boldsymbol{X} & I_{k(p-1)} \end{bmatrix} \begin{bmatrix} Y_0^{-1} & 0 \\ 0 & C^{(p-1)} \end{bmatrix} \begin{bmatrix} Y_0 & \boldsymbol{Y} \\ 0 & I_{k(p-1)} \end{bmatrix} \\ &= \begin{bmatrix} Y_0 & \boldsymbol{Y} \\ \boldsymbol{X} & C^{(p-1)} + \boldsymbol{X}Y_0^{-1}\boldsymbol{Y} \end{bmatrix}. \end{aligned} \tag{1.12}$$

eq:1.12

Equations (1.11) and (1.12) imply

$$\begin{cases} C_{i,j}^{(p)} = C_{i,j}^{(p-1)} + Z_{-(p-i)}W_0^{-1}W_{p-j}, & i,j = 1, \cdots, p-1, \\ C_{i,j}^{(p)} = C_{i-1,j-1}^{(p-1)} + X_{i-1}Y_0^{-1}W_{j-1}, & i,j = 2, \cdots, p. \end{cases} \tag{1.13}$$

eq:1.13

3. Inversion Formulae. Summing up either of (1.13) as a telescopic series, we easily find the following Gohberg-Semençul type inversion formulae:

$$C^{(p)} = \begin{bmatrix} X_0 & 0 & \cdots & \cdots & 0 \\ X_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ X_{p-1} & \cdots & \cdots & X_1 & X_0 \end{bmatrix} X_0^{-1} \begin{bmatrix} Y_0 & Y_{-1} & \cdots & \cdots & Y_{-(p-1)} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & Y_{-1} \\ 0 & \cdots & \cdots & \cdots & Y_0 \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ Z_{-(p-1)} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ Z_{-1} & \cdots & \cdots & Z_{-(p-1)} & 0 \end{bmatrix} Z_0^{-1} \begin{bmatrix} 0 & W_{p-1} & \cdots & \cdots & W_1 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & W_{p-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{1.14}$$

eq:1.14

and

$$C^{(p)} = \begin{bmatrix} Z_0 & Z_{-1} & \cdots & \cdots & Z_{-(p-1)} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & Z_{-1} \\ 0 & \cdots & \cdots & \cdots & Z_0 \end{bmatrix} Z_0^{-1} \begin{bmatrix} W_0 & 0 & \cdots & \cdots & 0 \\ W_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ W_{p-1} & \cdots & \cdots & W_1 & W_0 \end{bmatrix}$$
$$- \begin{bmatrix} 0 & X_{p-1} & \cdots & \cdots & X_1 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & X_{p-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} X_0^{-1} \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ Y_{-(p-1)} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ Y_{-1} & \cdots & \cdots & Y_{-(p-1)} & 0 \end{bmatrix}. \tag{1.15}$$

eq:1.15

4. Schur Complement Expressions. Let us now partition $A^{(p)}$ as follows:

$$A^{(p)} = \begin{bmatrix} H_0 & \check{H}_- \\ \check{H}_+ & A^{(p-1)} \end{bmatrix}, \qquad \check{H}_+ = \begin{bmatrix} H_1 \\ \vdots \\ H_{p-1} \end{bmatrix}, \qquad \check{H}_- = \begin{bmatrix} H_{-1} & \cdots & H_{-(p-1)} \end{bmatrix}.$$

Then

$$C^{(p)} = [A^{(p)}]^{-1} = \begin{bmatrix} (H_0^{\#})^{-1} & -(H_0^{\#})^{-1}\check{H}_- C^{(p-1)} \\ -C^{(p-1)}\check{H}_+(H_0^{\#})^{-1} & C^{(p-1)} + C^{(p-1)}\check{H}_+(H_0^{\#})^{-1}\check{H}_- C^{(p-1)} \end{bmatrix},$$

where

$$H_0^{\#} = H_0 - \check{H}_- C^{(p-1)}\check{H}_+.$$

Therefore,

$$X_0 = Y_0 = (H_0^{\#})^{-1},\ \boldsymbol{X} = -C^{(p-1)}\check{H}_+(H_0^{\#})^{-1},\ \boldsymbol{Y} = -(H_0^{\#})^{-1}\check{H}_- C^{(p-1)}. \tag{1.16}$$

eq:1.16

Analogously, let us now partition $A^{(p)}$ as follows:

$$A^{(p)} = \begin{bmatrix} A^{(p-1)} & \hat{H}_- \\ \hat{H}_+ & H_0 \end{bmatrix}, \qquad \hat{H}_- = \begin{bmatrix} H_{-(p-1)} \\ \vdots \\ H_{-1} \end{bmatrix}, \qquad \hat{H}_+ = \begin{bmatrix} H_{p-1} & \cdots & H_1 \end{bmatrix}.$$

Then

$$C^{(p)} = [A^{(p)}]^{-1} = \begin{bmatrix} C^{(p-1)} + C^{(p-1)}\hat{H}_-(H_0^{\times})^{-1}\hat{H}_+ C^{(p-1)} & -C^{(p-1)}\hat{H}_-(H_0^{\times})^{-1} \\ -(H_0^{\times})^{-1}\hat{H}_+ C^{(p-1)} & (H_0^{\times})^{-1} \end{bmatrix},$$

where

$$H_0^{\times} = H_0 - \hat{H}_+ C^{(p-1)}\hat{H}_-.$$

Therefore,

$$Z_0 = W_0 = (H_0^{\times})^{-1},\ \boldsymbol{Z} = -C^{(p-1)}\hat{H}_-(H_0^{\times})^{-1},\ \boldsymbol{W} = -(H_0^{\times})^{-1}\hat{H}_+ C^{(p-1)}. \tag{1.17}$$

eq:1.17

5. Algorithms. Let us write down the algorithm to compute $C^{(p)}$ from $C^{(p-1)}$.

a. Solve the linear system

$$A^{(p-1)} \begin{bmatrix} \check{X}_1 \\ \vdots \\ \check{X}_{p-1} \end{bmatrix} = \begin{bmatrix} H_1 \\ \vdots \\ H_{p-1} \end{bmatrix}.$$

b. Put $H_0^{\#} = H_0 - \sum_{s=1}^{p-1} H_{-s}\check{X}_s$.

c. Compute $X_0 = Y_0 = (H_0^{\#})^{-1}$.

d. Solve the linear system

$$\begin{bmatrix}\check{Y}_{-1} & \cdots & \check{Y}_{-(p-1)}\end{bmatrix} A^{(p-1)} = \begin{bmatrix}H_{-1} & \cdots & H_{-(p-1)}\end{bmatrix}.$$

e. Alternatively, put $H_0^{\#} = H_0 - \sum_{s=1}^{p-1} \check{Y}_{-s} H_s$.

f. Put

$$C_{i,j}^{(p)} = \begin{cases} C_{i-1,j-1}^{(p-1)} + \check{X}_{i-1} Y_0 \check{Y}_{j-1}, & i,j = 2, \cdots, p, \\ -\check{X}_{i-1} X_0, & i = 1, \cdots, p, \ j = 1, \\ -Y_0 \check{Y}_{-(j-1)}, & i = 1, \ j = 1, \cdots, p. \end{cases}$$

Alternatively, let us write down the algorithm to compute $C^{(p)}$ from $C^{(p-1)}$.

a. Solve the linear system

$$A^{(p-1)} \begin{bmatrix} \hat{Z}_{-(p-1)} \\ \vdots \\ \hat{Z}_{-1} \end{bmatrix} = \begin{bmatrix} H_{-(p-1)} \\ \vdots \\ H_{-1} \end{bmatrix}.$$

b. Put $H_0^{\times} = H_0 - \sum_{s=1}^{p-1} H_s \hat{Z}_{-s}$.

c. Compute $Z_0 = W_0 = (H_0^{\times})^{-1}$.

d. Solve the linear system

$$\begin{bmatrix}\hat{W}_{p-1} & \cdots & \hat{W}_1\end{bmatrix} A^{(p-1)} = \begin{bmatrix}H_{p-1} & \cdots & H_1\end{bmatrix}.$$

e. Alternatively, put $H_0^{\times} = H_0 - \sum_{s=1}^{p-1} \hat{W}_s H_{-s}$.

f. Put

$$C_{i,j}^{(p)} = \begin{cases} C_{i,j}^{(p-1)} + \hat{Z}_{-(p-i)} W_0 \hat{W}_{p-j}, & i,j = 1, \cdots, p-1, \\ -\hat{Z}_{-(p-i)} Z_0, & i = 1, \cdots, p, \ j = p, \\ -W_0 \hat{W}_{-(p-j)}, & i = p, \ j = 1, \cdots, p. \end{cases}$$

2 Solving Systems of Convolution Equations on Finite Intervals

sec:2

1. Auxiliary systems of convolution equations. Let $a > 0$ be finite and let $k \in L^1_{m\times m}(-a, a)$, i.e., k can be considered as an $m \times m$ matrix whose elements $k_{ij} \in$

$L^1(-a,a)$. Consider the following systems of convolution integral equations:

$$x(t) - \int_0^a k(t-s)x(s)\,ds = k(t), \qquad 0 \le t \le a, \qquad (2.1) \quad \boxed{\texttt{eq:2.1}}$$

$$z(-t) - \int_0^a k(s-t)z(-s)\,ds = k(-t), \qquad 0 \le t \le a, \qquad (2.2) \quad \boxed{\texttt{eq:2.2}}$$

$$w(t) - \int_0^a w(s)k(t-s)\,ds = k(t), \qquad 0 \le t \le a, \qquad (2.3) \quad \boxed{\texttt{eq:2.3}}$$

$$y(-t) - \int_0^a y(-s)k(s-t)\,ds = k(-t), \qquad 0 \le t \le a, \qquad (2.4) \quad \boxed{\texttt{eq:2.4}}$$

whose solutions $x(t)$ and $w(t)$ are sought in $\in L^1_{m\times m}(0,a)$ and $z(-t)$ and $y(-t)$ are sought in $\in L^1_{m\times m}(-a,0)$. Replacing t and s by $t-a$ and $s-a$, Eqs. (2.2) and (2.4) can also be written in the form

$$z(t-a) - \int_0^a k(t-s)z(s-a)\,ds = k(t-a), \qquad 0 \le t \le a, \qquad (2.5) \quad \boxed{\texttt{eq:2.5}}$$

$$y(t-a) - \int_0^a y(s-a)k(t-s)\,ds = k(t-a), \qquad 0 \le t \le a. \qquad (2.6) \quad \boxed{\texttt{eq:2.6}}$$

2. Main result. Let us search for a matrix function $\Gamma(t,s)$ ($0 \le t,s \le a$) which satisfies the system of convolution integral equations

$$\Gamma(t,s) - \int_0^a k(t-\tau)\Gamma(\tau,s)\,d\tau = k(t-s), \qquad 0 \le t,s \le a. \qquad (2.7) \quad \boxed{\texttt{eq:2.7}}$$

For $0 \le t \le a$ we easily compute

$$\begin{aligned} &g(t) + \int_0^a \Gamma(t,s)g(s)\,ds \\ &= g(t) + \int_0^a \left[k(t-s) + \int_0^a k(t-\tau)\Gamma(\tau,s)\,d\tau\right] g(s)\,ds \\ &= g(t) + \int_0^a k(t-s)\left[g(s) + \int_0^a \Gamma(s,\tau)g(\tau)\,d\tau\right] ds, \end{aligned}$$

so that

$$f(t) = g(t) + \int_0^a \Gamma(t,s)g(s)\,ds, \qquad 0 \le t \le a, \qquad (2.8) \quad \boxed{\texttt{eq:2.8}}$$

is a solution of Eq. (1.1). In other words, $\Gamma(t,s)$ is the resolvent kernel of the integral equation (1.1).

th:2.1 **Theorem 2.1** *Suppose the four equations* (2.1)-(2.4) *have a solution. Then for every* $g \in L^p_m(0,a)$ *the convolution equation* (1.1) *has a unique solution* $f \in$

$L^p_m(0,a)$ which is given by (2.8). *The resolvent kernel $\Gamma(t,s)$ is given by either of the two expressions*

$$\Gamma(t,s) = \begin{cases} x(t-s) + \int_0^s [x(t-\tau)y(\tau-s) - z(t-\tau-a)w(\tau-s+a)]\,d\tau, & s<t, \\ [3mm] y(t-s) + \int_0^t [x(t-\tau)y(\tau-s) - z(t-\tau-a)w(\tau-s+a)]\,d\tau, & t<s, \end{cases} \tag{2.9}$$

eq:2.9

and

$$\Gamma(t,s) = \begin{cases} w(t-s) + \int_t^a [z(t-\tau)w(\tau-s) - x(t-\tau+a)y(\tau-s-a)]\,d\tau, & s<t, \\ [3mm] z(t-s) + \int_s^a [z(t-\tau)w(\tau-s) - x(t-\tau+a)y(\tau-s-a)]\,d\tau, & t<s. \end{cases} \tag{2.10}$$

eq:2.10

Proof. It is clear that $\Gamma(t,s)$ can be viewed as an extension of the functions $x(t)$, $z(-t)$, $w(t)$, and $y(-t)$. One obviously has

$$\Gamma(t,0) = x(t), \qquad \Gamma(t,a) = z(t-a), \tag{2.11}$$

eq:2.12

$$\Gamma(0,s) = y(-s), \qquad \Gamma(a,s) = w(a-s). \tag{2.12}$$

eq:2.13

so that $\Gamma(t,s)$ is known on the boundary of the square $[0,a]\times[0,a]$.

Let us prove that

$$\Gamma(t,s) = X(t-s) + \chi_1(t,s) - \chi_2(t,s), \tag{2.13}$$

eq:2.14

where

$$X(u) = \begin{cases} x(u), & 0<u\le a, \\ y(u), & -a\le u<0, \end{cases}$$

$$\chi_1(t,s) = \int_0^{\min(t,s)} x(t-\tau)y(\tau-s)\,d\tau,$$

$$\chi_2(t,s) = \int_0^{\min(t,s)} z(t-\tau-a)w(\tau-s+a)\,d\tau.$$

Then for $\mu = \min(t, s)$ we have

$$\begin{aligned}
\chi_1(t,s) &= \int_0^{\mu} \left[k(t-\tau) + \int_0^{a} k(t-\tau-\hat{\tau}) x(\hat{\tau})\, d\hat{\tau} \right] y(\tau - s)\, d\tau \\
&= \int_0^{\mu} k(t-\tau) y(\tau - s)\, d\tau + \int_0^{\mu} \int_{\tau}^{a+\tau} k(t-u) x(u-\tau)\, du\, y(\tau - s)\, d\tau \\
&= \int_0^{\mu} k(t-\tau) y(\tau - s)\, d\tau + \int_0^{a} k(t-u) \int_0^{u} x(u-\tau) y(\tau - s)\, d\tau\, du \\
&+ \int_a^{a+\mu} k(t-u) \int_u^{a+\mu} x(u-\tau) y(\tau - s)\, d\tau\, ds.
\end{aligned}$$

□

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