TEOREMA. Sia $f \in C^{n+1}[x, x+h]$. Allora

$$f(x+h) = \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!} h^j + \int_x^{x+h} \frac{(x+h-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

D'altra parte, se $f \in C^{n+1}[x-h, x]$, allora

$$f(x-h) = \sum_{j=0}^{n} \frac{(-1)^j f^{(j)}(x)}{j!} h^j - \int_{x-h}^{x} \frac{(x+h-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

In particolare, se $f \in C^4[x-h, x+h]$, allora

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \tfrac{1}{6}h^3 f^{(3)}(x) + \tfrac{1}{6}\int_x^{x+h} (x+h-t)^3 f^{(4)}(t)\, dt,$$

$$f(x-h) = f(x) - hf'(x) + \tfrac{1}{2}h^2 f''(x) - \tfrac{1}{6}h^3 f^{(3)}(x) - \tfrac{1}{6}\int_{x-h}^{x} (x-h-t)^3 f^{(4)}(t)\, dt.$$

Ciò implica:

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) = \frac{1}{6h^2}\int_x^{x+h} (x+h-t)^3 f^{(4)}(t)\, dt - \frac{1}{6h^2}\int_{x-h}^{x} (x-h-t)^3 f^{(4)}(t)\, dt.$$

Quindi

$$\left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| \leq \tfrac{1}{24} h^2 \|f^{(4)}\|_\infty.$$

Nello stesso modo:

$$\begin{aligned}\frac{f(x+h)-f(x-h)}{2h}-f'(x)=\frac{1}{4h}&\int_x^{x+h}(x+h-t)^2f^{(3)}(t)\,dt\\&-\frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2f^{(3)}(t)\,dt.\end{aligned}$$

Quindi

$$\left|\frac{f(x+h)-f(x-h)}{2h}-f'(x)\right|\leq\tfrac{1}{12}h^2\|f^{(3)}\|_\infty.$$

Inoltre,

$$\frac{f(x+h)-f(x)}{h}-f'(x)=\frac{1}{2h}\int_x^{x+h}(x+h-t)f''(t),$$
$$\frac{f(x+h)-f(x)}{h}-f'(x)=\frac{1}{2h}\int_x^{x+h}(x+h-t)f''(t),$$

e dunque

$$\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right|\leq\frac{1}{2}h\|f''\|_\infty,$$
$$\left|\frac{f(x)-f(x-h)}{h}-f'(x)\right|\leq\frac{1}{2}h\|f''\|_\infty.$$