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Integrable nonlinear partial differential equations and applications

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Chapter 1

The material contained in the following pages is based on of the lectures given by Prof. T. Aktosun during the period 01/02/2010-17/02/2010 at the University of Cagliari in the context of the project "Program Visiting Professor," supported by Regione Autonoma della Sardegna.

Lecture 1: Introduction

In this lecture my main goal is the introduction of the Inverse Scattering Method (IST), i.e. a powerful method which allows us to solve some *integrable* nonlinear partial differential equation (NPDEs). In order to make clear this method, I start recalling the basic terminology and the history related to the integrability . The following items are analyzed:

- Nonlinear PDEs arising in description of water waves starting from 1850s;
- Some of these equations are *integrable* and some are not. We are only interested in the integrable ones;
- Scaling in DE, for mathematical analysis.
  For example, let us consider the Schrödinger equation

  $$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

  where $\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-27}$ erg-sec is the reduced Planck constant and $m = 9.1 \times 10^{-28}$ gm is the electron's mass. Putting

  $$\begin{cases} \widetilde{t} &= \hbar t \\ \widetilde{x} &= \frac{\sqrt{2m}}{\hbar}x \end{cases}$$

  we get the "usual" Schrödinger equation: $-\nabla^2_{\widetilde{x}}\psi + V\psi = i\frac{\partial\psi}{\partial\widetilde{t}}$.
  Another illustrative example is the Korteweg de-Vries (KdV) equation. In regard to physical applications this equation is written as:

  $$\frac{\partial\eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\eta\frac{\partial\eta}{\partial x} + \frac{2}{3}\frac{\partial\eta}{\partial x} + \frac{1}{3}\frac{\partial^3\eta}{\partial x^3}\right).$$

  Usually the mathematicians rescaling this equation in such a way that it is appear in the simplified form

  $$u_t - 6uu_x + u_x + u_{xxx} = 0,$$

  where $u(x,t) = -\eta(x,t)$.
  The previous examples show the advantage of the rescaling: the coefficients in the

rescaled equation will be simple. Normally scaling involves multiplying those variable by “positive” constants, but the multiplication by “negative” constants is accepted as well;

- No general theory to solve NPDEs exists, each must be studied separately;
- Interest in solving NPDEs, due to their phisical importance;
- Finding “exact solutions” in terms of elementary functions: such solutions may be used as test functions to test numerical methods being developed;
- Certain NPDEs (the initial value problem (IVP) for certain NPDEs) seem to be solvable by the so-called IST (proposed for the first time in 1967 by Gardner, Greene, Kruskal, and Miura) and such equations are known as integrable;
- There are two types of NPDEs: the integrable NPDE and non integrable NPDE. It is natural to ask what determines the integrability. The answer is not yet known, but we emphasize various common properties of “integrable” NPDEs help us to better understand integrability criteria;
- Conservation law: some integrable NPDE, can be written in the form

$$(\,)_t + \nabla\cdot(\,) = 0.$$

The term “conservation law” is appropriate because integrating the last equation

$$\iiint \left[(\xi)_t + \nabla\cdot(\vec{F})\right] = \frac{d}{dt}\iiint(\xi) + \oiint \vec{F}\cdot d\vec{a} = 0$$

and taking into account that as the surface goes to infinity the second term in the last equation may vanish, we get $\frac{d}{dt}\iiint(\xi) = 0$, implying the conserved quantities. Many integrable NPDE have many conserved quantities.

History related to integrability

1834, Russell's observation of a solitary water wave at Union Canal;
1844, Russell's report to the British for AS;
1877, Boussinesq's book includes the NPDE, later called the KdV equation;
1895, de-Vries PhD thesis, Korteweg de-Vries paper which contains the KdV equation and its special solution in the form $\frac{1}{\cosh^2}$;
1954, Fermi-Pasta-Ulam puzzle (the object studied in this paper was related 64 masses interconnected with nonlinear springs and the analysis of the equipartition to the energy);
1965, Zabusky and Kruskal in your publication coined the phrase “soliton”, i.e. solutions to the KdV equation with nonlinear interactions, and explaining the solution to the Fermi-Pasta-Ulam puzzle based on a numerical solution to KdV equation;
1967, Gardner, Green, Kruskal, Miura solved the initial value problem for KdV by Inverse Scattering transform (IST);
1968, Lax's method to derive integrable NPDE;
1972, Zakharov-Shabat used IST to solve nonlinear Schrödinger equation (NLS);
1972, Wadati used IST to solve the modified KdV equation (mKdV) via IST (Wadati's work was one-page long);
1973 Ablowitz, Kaupp, Newell, Segur (AKNS) used the IST to solve the sine-Gordon equation (SG).

Example of integrable NPDEs

There are two types of integrable NPDEs:

1. KdV-like equation: contain the KdV equation and higher order hierarchies;

2. NLS-like equation: contain the NLS equation, the mKdV equation, the SG equation and higher order hierarchies.

This classification is related to the corresponding linear ODE. For example the NLS, mKdV and SG equation are all related to the first order linear system

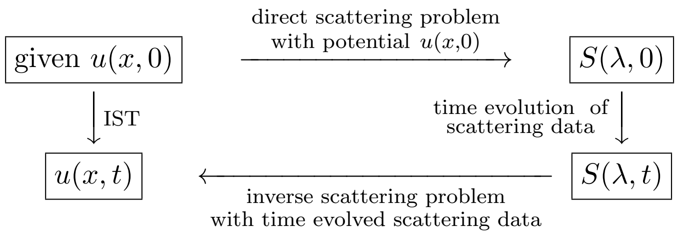
$$\begin{cases} \xi' &= -i\lambda\xi + q\eta \\ \eta' &= v\xi + i\lambda\eta \end{cases}$$

for various choices of v in relation to u (e.g. $q = u, v = -u^*$ give us the *focusing NLS*, choosing $q = u, v = u^*$ we have the *defocusing NLS*, instead taking $q = -\dfrac{u_x}{2}, v = \dfrac{u_x}{2}$ we get the SG equation). In the Table 1 -at the end of this section- the named integrable equation and their ODE associated are displayed.

The name NLS arises because the linear part of this equation coincide with the Schrödinger equation. It is also important to explain the difference between "focusing" and "defocusing" NLS equation. In the focusing case the nonlinearity tries to focus the solution, the dispersion tries to broaden the solution. Then these two effects work against each other, and, in case of exact balance, we get a soliton solution. In the defocusing case nonlinearity and dispersion both work to broaden the solutions, hence in general we cannot expect soliton solutions in this case.

Basic idea behind IST

To solve the initial value problem for an integrable NPDE do the following



More precisely, we have to do the following step:

1. Associate the NPDE with a LODE;

2. Associate the initial value $u(x,0)$ for NPDE with the potential (i.e. a coefficient term) in the LODE;

3. Exploit the 1-1 correspondence between the potential $u(x,0)$ in the LODE and the scattering data $S(\lambda,0)$, which is related to the spatial asymptotics (as $x \to +\infty$ or $x \to -\infty$) of a special "scattering solution" $\Psi(\lambda,x,0)$. Let us remark the basic idea: there should be 1-1 correspondence between $u(x,0)$ and $S(\lambda,0)$. Usually the parameter count is a good guide to guess the 1-1 association. For example, one single variable x in $u(x,0)$ correspond to one single parameter λ in $S(\lambda,0)$;

4. "Time evolution" of the scattering data from $S(\lambda,0)$ to $S(\lambda,t)$, usually by multiplying a single phase factor, e.g. e^{8ik^3t} for KdV equation, $e^{-4i\lambda^2t}$ for NLS equation. This will be studied in more detail for each specific integrable NPDE;

5. Exploit the 1-1 correspondence for eaach fixed t between $u(x,t)$ and $S(\lambda,t)$. The determination $S(\lambda,t) \mapsto u(x,t)$ is known as solving the inverse problem for the corresponding LODE at time t. (The determination $u(\lambda,t) \mapsto S(\lambda,t)$ correspond to solving the direct problem for the LODE).

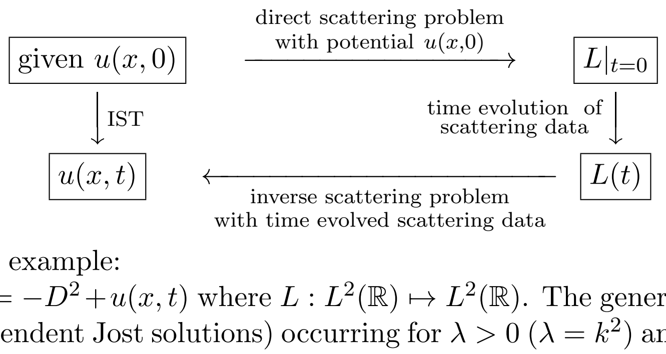
6. It is amazing and surprising that $u(x,t)$ solves the NPDE and $\lim_{t\to 0} u(x,t)$ corresponds to the initial value $u(x,0)$.

A common attack to look for special solutions to NPDE

Consider a "wave" solution in the form $u = f(x - ct)$ and try to determine the wavespeed c and the profile function f. One advantage of this is that f satisfies a (nonlinear) ODE even though u must satisfy a NPDE. This simplification at times works, e.g. for the KdV equation where $f \sim \frac{1}{\cosh^2}$. One can try, e.g. $u(x,t) = \alpha \frac{1}{\cosh^2(\beta x + \gamma t + \varepsilon)}$ for some constant parameters $\alpha, \beta, \gamma, \varepsilon$. Now one can try to determine the relationships among such parameters to get a solution to KdV equation.

Lecture 2 and 3

We can see the IST as a method which associate an integrable NPDE for $u(x,t)$ to a linear ODE $L\Psi = \lambda\Psi$ with a spectral parameter λ where $u(x,t)$ appear as a coefficient in the linear differential operator. It is necessary to analyze the spectrum of L and determine a scattering-spectral data set for L that uniquely determines L. Various mathematical problems should be investigated. For example, what is the domain of L (i.e. what function space does L act on?). Depending on $u(x,t)$, L may have continuous spectrum, discrete spectrum, singular spectrum, etc. By putting restrictions on the function class to which $u(x,t)$ belongs to, the spectrum of L can be made more manageable (e.g. the "spectral singularities" or discrete eigenvalues imbedded in the continuous spectrum can be avoided by using "nicer" $u(x,t)$). Now we can reformulate the basic idea of the IST as follows: In order to solve the initial value problem for the integrable NPDE starting from $u(x,0)$, analyze the spectral scattering properties of the linear operator L at $t = 0$. Or equivalently, analyze the "standard" and "generalized" eigenvectors (By standard eigenvalues and standard eigenvectors we mean eigenvectors of "finite length", e.g. $\|\Psi\|_{L^2}$ is finite or $\|\Psi\|_{L^1}$ is finite). Such standard eigenvectors correspond to bound-states of the linear operator L. Generalized eigenvectors correspond to "bounded" solutions to $L\Psi = \lambda\Psi$ and in the physics literature they are usually known as the scattering states. If $u(x,t)$ for each fixed "t" belongs to a "nice" class (such as $u(\cdot,t) \in L^1(\mathbb{R})$ for the 1-D Schrödinger generator $L = -\frac{d^2}{dx^2} + u(x,t)$) then it can be shown that the spectrum of L consists of all $\lambda > 0$ and at most a finite number of dicrete (negative) λ values. The case $\lambda = 0$ must be analyzed separately because in that case it may or may not be possible to have a bounded solution; for example in the so-called exceptional case there is a bounded solution but in the generic case the two linearly indepedent solutions at $\lambda = 0$ are both unbounded in x. So the scheme of IST is represented in the following diagram:



Let us discuss this example:

Let us consider $L = -D^2 + u(x,t)$ where $L : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$. The generalized eigenvectors (two linearly independent Jost solutions) occurring for $\lambda > 0$ ($\lambda = k^2$) and their asymptotic behaviour is as below

$$f_l(k,x,t) \sim e^{ikx}, \qquad x \to +\infty$$
$$f_r(k,x,t) \sim e^{-ikx}, \quad x \to -\infty.$$

If $u(\cdot,t) \in L^1_1(\mathbb{R})$, i.e. $\int_{-\infty}^{\infty} dx\,(1+|x|)\,|u(x,t)|$ is finite, then $f_l(\cdot,x,t)$ and $f_r(\cdot,x,t)$ have analytic extensions for $k \in \mathbb{R}$ to $k \in \mathbb{C}^+$ with asymptotics in k as $f_l(\cdot,x,t) \sim e^{ikx}$ and

Name	NPDE	LODE	Lax pair	AKNS
KdV	$u_t - 6uu_x + u_{xxx} = 0$	$(-\frac{d^2}{dx^2} + u)\Psi = k^2\Psi$	$L = -\partial_x^2 + u(x,t)$ $A = -4\partial_x^3 + 6u\partial_x + 3u_x$	$\mathcal{X} = \begin{pmatrix} 0 & u(x,t)-\lambda \\ 1 & 0 \end{pmatrix}$ $\mathcal{T} = \begin{pmatrix} u_x & -4\lambda^2+2\lambda u+2u^2-u_{xx} \\ 4\lambda+2u & -u_x \end{pmatrix}$
NLS	$iu_t + u_{xx} + 2\|u\|^2 u = 0$	$\begin{cases} \xi' &= -i\lambda\xi + u(x,t)\eta \\ \eta' &= i\lambda\eta - \overline{u(x,t)}\xi \end{cases}$	$L = \begin{pmatrix} i\partial_x & -iu(x,t) \\ -i\overline{u(x,t)} & -i\partial_x \end{pmatrix}$ $A = \begin{pmatrix} 2i\partial_x^2+i\|u\|^2 & -2iu-iu_x \\ -2i\overline{u}-i\overline{u}_x & -2i\partial_x^2-i\|u\|^2 \end{pmatrix}$	$\mathcal{X} = \begin{pmatrix} -i\lambda & u(x,t) \\ -\overline{u(x,t)} & i\lambda \end{pmatrix}$ $\mathcal{T} = \begin{pmatrix} -2i\lambda^2+i\|u\|^2 & 2\lambda u+iu_x \\ -2\lambda\overline{u}+i\overline{u}_x & 2i\lambda^2-i\|u\|^2 \end{pmatrix}$
sine-Gordon	$u_{xt} = \sin u$	$\begin{cases} \xi' &= -i\lambda\xi - \frac{u_x}{2}\eta \\ \eta' &= \frac{u_x}{2}\xi + i\lambda\eta \end{cases}$	$L = \begin{pmatrix} i\partial_x & \frac{iu_x}{2} \\ \frac{iu_x}{2} & -i\partial_x \end{pmatrix}$ $A = \frac{1}{8}\begin{pmatrix} C\partial_x^{-1}C-S\partial_x^{-1}S & -S\partial_x^{-1}C-C\partial_x^{-1}S \\ S\partial_x^{-1}C+C\partial_x^{-1}S & C\partial_x^{-1}C-S\partial_x^{-1}S \end{pmatrix}$	$\mathcal{X} = \begin{pmatrix} -i\lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & i\lambda \end{pmatrix}$ $\mathcal{T} = \frac{i}{4\lambda}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}$
X Dym	$u_t = u^3 u_{xxx}$		$L = u^2\frac{d^2}{dx^2}$ $A = 4u^3\partial_x^3 + 6u^2u_x\partial_x^2$	$\mathcal{X} = \begin{bmatrix} 0 & \frac{\lambda}{u^2} \\ 1 & 0 \end{bmatrix}$ $\mathcal{T} = \begin{bmatrix} 2\lambda u_x & 4\frac{\lambda^2}{u} - 2\lambda u_{xx} \\ 4\lambda u & -2\lambda u_x \end{bmatrix}$
Degasperis-Procesi	$u_t - u_{xxt} + 2\kappa u_x + 4uu_x$ $= 3u_xu_{xx} + uu_{xxx}$		$L = \frac{1}{m+\frac{2}{3}k}(\partial_x^3 - \partial_x)$ $A = \partial_x^2(\partial_x^3 - \partial_x)^{-1}(m + \frac{2}{3}k) - u\partial_x + u_x$	$\mathcal{X} = \begin{bmatrix} 0 & 1 & (m+\frac{2}{3}k)\lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\mathcal{T} = \begin{bmatrix} \frac{1}{\lambda} - u_x & \frac{2}{3}k & u_x - \lambda u(m+\frac{2}{3}k) \\ -u & \frac{1}{\lambda} & u + \frac{2}{3}k \\ \frac{1}{\lambda} & -u & u_x \end{bmatrix}$
Kadomtsev-Petviashvili	$(u_t - 6uu_x + u_{xxx})_x = -3u_{yy}$		$L = -\partial_x - i\partial_y + u(x,y,t)$ $A = -4\partial_x^3 + 6u\partial_x + 3u_x - 3i\partial_x^{-1}u_y$	$\mathcal{X} = \frac{1}{\epsilon}\partial_x^2 + \frac{u-\lambda}{\epsilon}$ $\mathcal{T} = -4\partial_x^3 + 6u\partial_x + 3u_x - 3\epsilon w$

Table 1.1: Some integrable equations. For complete understooding of the notations used here see Lectures 4,5 and 6.

$f_r(\cdot, x, t) \sim e^{-ikx}$ as $k \to \infty$ in $\overline{\mathbb{C}}^+$, where $\overline{\mathbb{C}}^+ := \mathbb{C}^+ \cup \mathbb{R}$. At most a finite number of bound states occurs when $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ for $\lambda_j < 0$. It is convenient to use $k_j = \sqrt{\lambda_j} = i\kappa_j$ and use $f_l(i\kappa_j, x, t)$ or $f_r(i\kappa_j, x, t)$ as eigenvectors. It is important to remark that $f_l(i\kappa_j, x, t)$ and $f_r(i\kappa_j, x, t)$ are linearly dependent as functions of x, i.e. there is a "dependency constant $\gamma_j(t)$" that depends on t but not on x such that $f_l(i\kappa_j, x, t) = \gamma_j(t) f_r(i\kappa_j, x, t)$.

How to solve $u(x, 0) \longrightarrow S(\lambda, 0)$**.** Simply solve the linear ODE $L\Psi = \lambda\Psi$ and obtain standard and generalized eigenvalues and eigenvectors. Then, from the spatial asymptotics of those eigenvectors construct the spatial-scattering data that uniquely correspond to $u(x, 0)$.

How to solve $S(\lambda, 0) \longrightarrow S(\lambda, t)$**.** This time evolution is expected to be rather simple. Once the standard and generalized eigenvectors are known as function of t, then we can analyze for $\Psi(\lambda, x, 0) \longrightarrow \Psi(\lambda, x, t)$ with the help of the Lax method or the AKNS method. In general the time evolution of each Jost function is fairly complicated.

How to solve $S(\lambda, t) \longrightarrow u(\lambda, t)$**.** This is the final step in the IST and it is known as inverse scattering problem. Such an inverse scattering problem can be solved by the Marchenko method or by the Gel'fand-Levitan method or by some others methods. The Marchenko method consists of the solution to a suitable integral equation which involves an integration on the semi-infinite interval (x, ∞) and whose kernel is related to the Fourier transform of the scattering data. The solution of the Marchenko integral equation is related to the potential $u(x, t)$. The Gel'fand-Levitan method presents some little differences. In order to obtain the potential it is necessary to solve the so-called Gel'fand-Levitan integral equation which involves an integration on the finite interval $(0, x)$ and its kernel is related to the Fourier trasform of the spectral measure associated with the LODE.

Many authors try to solve the inverse scattering problem using the so-called Riemann-Hilbert problem. Let us see in which consists this problem. For the sake of convenience I introduce this problem in relation to the Schrödinger equation. Then we consider

$$(-\frac{d^2}{dx^2} + u(x))\Psi = k^2\Psi,$$

and recall that the behavior of the Jost is as follows

$$f_l(k, x) \sim e^{ikx}, \quad \text{as}\, x \to +\infty, \qquad f_l(k, x) \sim \frac{1}{T}e^{ikx} + \frac{L}{T}e^{-ikx}, \quad \text{as}\, x \to -\infty$$
$$f_r(k, x) \sim e^{-ikx}, \quad \text{as}\, x \to -\infty, \qquad f_r(k, x) \sim \frac{1}{T}e^{-ikx} + \frac{R}{T}e^{ikx}, \quad \text{as}\, x \to +\infty$$

It is well known in literature that $f_l(\cdot, x)$ has an analytic extension to $\mathrm{Im}k > 0$ and it is continuous to $\mathrm{Im}k \geq 0$. Now it is possible to define functions $g_l(k, x)$ and $g_r(k, x)$ as below

$$\begin{pmatrix} g_l(k, x) \\ g_r(k, x) \end{pmatrix} := \begin{pmatrix} f_l(-k, x) \\ f_r(-k, x) \end{pmatrix}$$

and the domain of $g_l(\cdot, x)$ and $g_r(\cdot, x)$ is $\mathrm{Im}k \leq 0$. In the common k-domain of $f_l(k, x)$, $f_r(k, x)$, $g_l(k, x)$, $g_r(k, x)$ the following equation hold

$$g_l(k, x) = a(k) f_l(k, x) + b(k) f_r(k, x), \quad k \in \mathbb{R},$$
$$g_r(k, x) = c(k) f_l(k, x) + d(k) f_r(k, x), \quad k \in \mathbb{R}.$$

Exploiting the asymptotic behavior of $f_{l/r}(k, x)$ it is possible to determine the coefficients $a(k)$, $b(k)$, $c(k)$, and $d(k)$ in terms of the so-called scattering coefficients T (transmission coefficient), L and R (reflection coefficients from the left and right, respectively):

$$g_l(k, x) = T(k) f_l(k, x) - R(k) f_r(k, x)$$
$$g_r(k, x) = -L(k) f_l(k, x) + T(k) f_r(k, x).$$

The above equations relating functions analytic in $k \in \mathbb{C}^+$ and analytic in $\mathbb{C}^-$ form a Riemann-Hilbert problem. More generally, a Riemann-Hilbert problem can be introduced as follows: Let us consider two functions $f_+(k)$ and $f_-(k)$ being $f_+(k)$ ($f_-(k)$) analytic for $|z| > 1$ ($|z| < 1$) and continuous for $|z| \geq 1$ ($|z| \leq 1$) and suppose we specify the behavior of $f_+(k)$ as $k \to \infty$. In the common domain of $f_+(k), f_-(k)$, i.e, $|z| = 1$, we may have, e.g.

$$f_+(k) = S(k)f_-(k).$$

Solving the Riemann-Hilbert problem for $f_l(k)$ and $f_r(k)$ is equivalent to finding $S(k) \longrightarrow fl/r(k,x)$ and hence once $f_l(k)$ or $f_r(k)$ is determined, we can recover the potential $u(x)$ as

$$u(x) = \frac{k^2 f_l(k,x) + f_l^{''}(k,x)}{f_l(k,x)}.$$

Now let us discuss the following simple example: Suppose that $S(k) = \dfrac{k+i}{k-i}$. The Riemann-Hilbert problem reduces to the equation $f_+(k) = \dfrac{k+i}{k-i} f_-(k)$ where $f_+(k)$ (respectively, $f_-(k)$) is analytic in $Imk > 0$ ($Imk < 0$) and $f_+(k) \to 1$ as $k \to \infty$ in $Imk > 0$. We can write

$$(k-i)f_+(k) = (k+i)f_-(k) = F(k).$$

Since the common domain of $f_+(k)$ and $f_-(k)$ is the real axis, the function $f_+(k)$ is defined for $k \in \mathbb{R}$, but admit an analytic extension on the upper plane $\mathbb{C}^+$. Then $(k-i)f_+(k)$ has an analytic extension in $\mathbb{C}^+$. To the other hand, using similar arguments, we can conclude that $(k+i)f_-(k)$ has an analytic extension on $\mathbb{C}^-$. As a result the function $F(k)$ is an entire function in the form $F(k) = c_1 k + c_0(k)$. Taking into account the condition $f_+(k) \to 1$ as $k \to \infty$ we get

$$\begin{aligned} f_+(k) &= \frac{k + c_0(k)}{k-i} \\ f_-(k) &= \frac{1 + c_0(k)}{k+i}. \end{aligned}$$

Lectures 4, 5, 6

In these lectures I introduce the three main methods, i.e, Lax method, AKNS method and the alternative Lax method, actually used in order to establish the integrability of a given NPDE. These methods allow help to determine an integrable NPDE corresponding to a *given* LODE.

It is a very difficult task to determine if a NPDE is integrable or not (in the sense that the IST method is applicable or not). On the other hand, if we know that a LODE is associated with a NPDE, we can state its integrability and use one of these three methods to determine the corresponding NPDE.

Lax method

The first method that I will discuss is known as the **Lax method** (proposed the first time by P. Lax in 1968). This method consists of the introduction of two linear operators L and A such that

$$\begin{cases} L\Psi = \lambda\Psi, & \text{spatial evolution of } \Psi, \\ \Psi_t = A\Psi, & \text{time evolution of } \Psi, \end{cases}$$

where Ψ depends on λ, x, t. We can reformulate the Lax method as follows: Given L with $L\Psi = \lambda\Psi$, find a linear operator A such that $\Psi_t - A\Psi$ satisfies

$$L(\Psi_t - A\Psi) = \lambda(\Psi_t - A\Psi).$$

The last equation is equivalent to $L\Psi_t - LA\Psi = \lambda\Psi_t - A\lambda\Psi$, which can be written as $(L\Psi)_t - L_t\Psi - LA\Psi = \lambda\Psi_t - A\lambda\Psi$ and taking into account that $L\Psi_t = \lambda\Psi_t$ and $\lambda_t = 0$ we get $(L_t + LA - AL)\Psi = 0$, which implies $L_t + LA - AL = 0$. Even though L and A are differential operators, we require that $L_t + LA - AL$ should not be a differential operator but it should be a multiplication operator. Note that $L_t + LA - AL$ does not contain λ (this is a consequence of the fact that L and A are linear differential operators not containing λ). Then, the Lax method can be described as follows: Given L, find A such that

$$L_t + LA - AL = 0, \tag{1.0.1}$$

which will yield the NPDE.

Examples. Let us illustrate this method in several important cases.

- Let us consider the *KdV equation* $u_t - 6uu_x + u_{xxx} = 0$. We will prove that this equation is integrable. In order to obtain this result, we consider the linear ODE, which is the Schrödinger equation

  $$-\frac{d^2}{dx^2}\Psi + u(x,t)\Psi = \lambda\Psi.$$

  This equation can be written as $L\Psi = \lambda\Psi$ where

  $$L = -\frac{d^2}{dx^2} + u(x,t). \tag{1.0.2}$$

  Let us try to determine the associated operator A by assuming that it has the form

  $$A = \alpha_3\partial_x^3 + \alpha_2\partial_x^2 + \alpha_1\partial_x + \alpha_0, \tag{1.0.3}$$

where the coefficients α_j with $j = 0, 1, 2, 3$ may depend on x and t, but no on the spectral parameter λ. Using (1.0.2) and (1.0.3) in (1.0.1), we obtain

$$(\,)\partial_x^5 + (\,)\partial_x^4 + (\,)\partial_x^3 + (\,)\partial_x^2 + (\,)\partial_x + (\,) = 0. \tag{1.0.4}$$

Because of the operator $L_t + LA - AL$ is a multiplication operator, each coefficient denoted by ( ) must vanish. The coefficient of ∂_x^5 vanishes automatically. Setting the coefficients of ∂_x^j to zero for $j = 4, 3, 2, 1$, we get

$$\alpha_3 = c_1, \quad \alpha_2 = c_2, \quad \alpha_1 = c_3 - \frac{3}{2}c_1 u, \quad \alpha_0 = c_4 - \frac{3}{4}c_1 u_x - c_2 u,$$

with c_1, c_2, c_3, and c_4 denoting arbitrary constants. Making the above choices and putting $c_1 = -4$ and $c_3 = 0$, (1.0.4) becomes $u_t - 6uu_x + u_{xxx} = 0$, i.e. the KdV equation. Then we can state that the KdV equation arises from the compatibility condition (1.0.1), choosing $L = -\dfrac{d^2}{dx^2} + u(x,t)$ and $A = -4\partial_x^3 + 6u\partial_x + 3u_x$.

- Let us consider the first-order system

$$\frac{d}{dx}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -i\lambda & \frac{1}{2}u_x \\ -\frac{1}{2}u_x & i\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

This system suggests to choose L and A in the following form

$$L = \begin{bmatrix} i\partial_x & \frac{i}{2}u_x \\ \frac{i}{2}u_x & -i\partial_x \end{bmatrix},$$

$$A = \frac{1}{8}\begin{bmatrix} C\partial_x^{-1}C - S\partial_x^{-1}S & -S\partial_x^{-1}C - C\partial_x^{-1}S \\ S\partial_x^{-1}C + C\partial_x^{-1}S & C\partial_x^{-1}C - S\partial_x^{-1}S \end{bmatrix},$$

where

$$Cf = \cos\left(\frac{u}{2}\right)\cdot f, \quad Sf = \cos\left(\frac{u}{2}\right)\cdot f,$$

and

$$\partial_x^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right), \quad \partial_x^{-1}f(x,t) = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right) ds\, f(s,t).$$

If one imposes the condition (1.0.1), then one obtains the corresponding NPDE as $u_{xt} = \sin(u)$, i.e. the *sine-Gordon* equation.

- Let us consider the *Dym equation*:

$$u_t = u^3 u_{xxx}.$$

This equation is integrable because arises from (1.0.1) when the Lax pair L and A are chosen as

$$L = u^2\frac{d^2}{dx^2}, \quad A = 4u^3\partial_x^3 + 6u^2u_x\partial_x^2.$$

- Let us introduce the *Degasperis-Procesi* equation:

$$u_t - u_{xxt} + 2\kappa u_x + 4uu_x = 3u_xu_{xx} + uu_{xxx},$$

where κ is a constant. Putting $m := u - u_{xx}$ and taking L and A as

$$L = \frac{1}{m + \frac{2}{3}k}(\partial_x^3 - \partial_x), \quad A = \partial_x^2(\partial_x^3 - \partial_x)^{-1}(m + \frac{2}{3}k) - u\partial_x + u_x,$$

it is straightforward to see that the Degasperis-Procesi equation is integrable because it derived as in from (1.0.1) with the above specified choices for L and A.

- Let us now establish the integrability for another important equation, i.e. the *Kadomtsev-Petviashvili (KPI)* equation:

$$(u_t - 6uu_x + u_{xxx})_x = -3u_{yy},$$

where u is a function depending on x, y, t. It can be seen that the KPI equation can be written as

$$u_t - 6uu_x + u_{xxx} = -3\partial_x^{-1}u_{yy},$$

with $\partial_x^{-1} := \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right)$, i.e.

$$(\partial_x^{-1}f)(x, y, t) := \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) ds\, f(s, y, t).$$

Choosing L and A, respectively, as

$$L = -\partial_x - i\partial_y + u(x, y, t), \; A = -4\partial_x^3 + 6u\partial_x + 3u_x - 3i\partial_x^{-1}u_y,$$

and imposing (1.0.1), we find

$$(\,)\partial_x^5 + (\,)\partial_x^4 + (\,)\partial_x^3 + (\,)\partial_x^2 + (\,)\partial_x + (u_t - 6uu_x + u_{xxx} + 3\partial_x^{-1}u_{yy}) = 0.$$

Now we observe that each coefficient denoted by $(\,)$ automatically vanishes, and as a result the last equation give us

$$u_t - 6uu_x + u_{xxx} + 3\partial_x^{-1}u_{yy} = 0,$$

i.e. the KPI equation.

- All the equations considered until now have only one potential function $u(x, t)$. Now let us introduce an example in which two potentials u and v are involved:

$$\begin{cases} iq_t + q_{xx} - 2q^2r = 0, \\ ir_t + r_{xx} - 2qr^2 = 0. \end{cases}$$

Choosing the relationship between q and r we obtain various integrable equations from the previous system. For example putting $q = -\frac{1}{2}u_x$, $r = \frac{1}{2}u_x$ we obtain the sine-Gordon equation, choosing $q = - \mp r^* = u$ we obtain the NLS equation $iu_t + u_xx \pm 2|u|^2u = 0$. Another interesting choice is $q = \mp r$ which leads us to the modified KdV equation $u_t \pm 6u^2u_x + u_{xxx} = 0$. Let us develope the Lax method for the *NLS equation.* It is well known, as shown by Zakharov and Shabat in 1972 that this equation is related to the following linear system of differential equation

$$\frac{d}{dx}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -i\lambda & u \\ -u^* & i\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

The LODE associated with the NLS suggests to choose L as

$$L = \begin{bmatrix} i\frac{d}{dx} & -iu \\ -iu^* & -i\frac{d}{dx} \end{bmatrix},$$

where the dependence on t of L is confined only to the potential u in such a way that

$$L_t = \begin{bmatrix} 0 & -iu_t \\ -iu_t^* & 0 \end{bmatrix}.$$

Now we look for an operator A in the following form:

$$A = \begin{bmatrix} a_2\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0 & b_2\frac{d^2}{dx^2} + b_1\frac{d}{dx} + b_0 \\ c_2\frac{d^2}{dx^2} + c_1\frac{d}{dx} + c_0 & d_2\frac{d^2}{dx^2} + d_1\frac{d}{dx} + d_0 \end{bmatrix},$$

where the coefficients a_i, b_i, c_i, d_i for $i = 0$, 1, 2 do not depend on λ but they may depend on x and t. Now, putting $D := \dfrac{d}{dx}$ and recalling that the operator Df can be written as $Df = f' + fD$, we impose the condition

$$L_t + LA - AL = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

after some straightforward calculations, and with the choices

$$\begin{aligned} &a_2 = -d_2 = 2i, \quad a_1 = d_1 = 0, \quad a_0 = -d_0 = i|u|^2, \quad b_2 = c_2 = 0, \\ &b_1 = -2iu, \quad b_0 = -iu_x, \quad c_1 = -2iu^*, \quad b_0 = -iu_x^*, \end{aligned}$$

we obtain

$$L_t + LA - AL = \begin{bmatrix} 0 & iu_t + u_{xx} + 2|u|^2u \\ iu_t^* + u_{xx}^* + 2|u|^2u^* & 0 \end{bmatrix}.$$

As a result the (1.0.1) is satisfied if one chooses

$$L = \begin{bmatrix} iD & -iu \\ -iu^* & -iD \end{bmatrix}, \quad A = \begin{bmatrix} 2iD^2 + i|u|^2 & -2iuD - iu_x \\ -2iu^*D - iu_x^* & -2iD^2 - i|u|^2 \end{bmatrix},$$

and requires that

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad iu_t^* + u_{xx}^* + 2|u|^2u^* = 0,$$

i.e. the NLS equation holds.

- The final NPDE whose integrability I will demonstrate by the Lax method is the so-called *Jaulent equation:*

$$-\Psi_{xx} + u\Psi + kv\Psi = k^2\Psi,$$

where k is the spectral parameter. Putting

$$\Phi := \begin{bmatrix} \Psi \\ k\Psi \end{bmatrix},$$

we can introduce the linear operator L as

$$L = \begin{bmatrix} 0 & 1 \\ \partial_x^2 + u & v \end{bmatrix},$$

in such a way that we rewrite the Jaulent equation in the form $L\Phi = k\Phi$ as

$$\begin{bmatrix} 0 & 1 \\ \partial_x^2 + u & v \end{bmatrix} \begin{bmatrix} \Psi \\ k\Psi \end{bmatrix} = k \begin{bmatrix} \Psi \\ k\Psi \end{bmatrix}.$$

Now we have to determine the operator A in such a way that (1.0.1), i.e. $L_t + LA - AL = 0$ holds. It is possible to prove that A must be taken as

$$\begin{bmatrix} -4\partial_x^3 + (6u + \frac{3}{2}v^2)\partial_x + 3u_x - \frac{3}{2}vv_x & 6v\partial_x + 3v_x \\ -6v\partial_x^3 - 3v_x\partial_x^2 + 6u_xv + 3uv_x & -4\partial_x^3 + (6u + \frac{15}{2}v^2)\partial_x + 3u_x - \frac{15}{2}vv_x \end{bmatrix}.$$

For the sake of brevity we have specified the operator A, which yields the NPDE integrable associated with the Jaulent equation as

$$\begin{cases} u_t - 6uu_x + u_{xxx} + \dfrac{3}{2}vv_{xxx} - \dfrac{9}{2}v_xv_{xx} - 6uvv_x - \dfrac{3}{2}u_xv^2 = 0, \\ v_t + v_{xxx} - 6uv_x - 6u_xv - \dfrac{15}{2}v^2v_x = 0. \end{cases}$$

AKNS method

Another powerful method which permits to determine an integrable NPDE corresponding to a given LODE is the AKNS method. This method was used the first time by Ablowitz, Kaup, Newell, and Segur in 1973 in order to establish the integrability to the sine-Gordon equation. The basic idea behind this method is the following: write $L\Psi = \lambda\Psi$ as a first-order system in the form $\theta_x = X\theta$ and find an operator T such that $\theta_t - T\theta$ satisfies

$$(\theta_t - T\theta)_x = X(\theta_t - T\theta),$$

(the operators X and T are said to form an AKNS pair). From the last equation we get

$$\theta_{tx} - T_x\theta - T\theta_x = X\theta_t - XT\theta. \tag{1.0.5}$$

Taking into account that $\theta_{tx} = (\theta_x)_t = (X\theta)_t = X_t\theta + X\theta_t$, the equation (1.0.6) can be written as

$$X_t\theta + X\theta_t - T_x\theta - TX\theta = X\theta_t - XT\theta,$$

then after suitable simplications we arrive at the equation

$$(X_t - T_x + XT - TX)\,\theta = 0.$$

Since we choose X and T as multiplication operators (not differential operator), the operator $X_t - T_x + XT - TX$ is a multiplication operator, i.e. is not a differential operator. This implies

$$X_t - T_x + XT - TX = 0, \tag{1.0.6}$$

where the operator X contains the spectral parameter λ, and hence T also depend on λ as well. Summarizing the AKNS method can be illustrated as follows: Given X with $\theta_x = X\theta$, find T such that (1.0.6) is satisfied. The NPDE equation coming from to (1.0.6) is integrable; i.e., this equation is solvable with the help of the solutions to the direct and inverse scattering problems for the linear systems $\theta_x = X\theta$. Now I will discuss several examples in order to illustrate this method.

Examples.

- Let us consider the KdV equation: $u_t - 6uu_x + u_{xxx} = 0$. It is well-known that the LODE corresponding to this equation is the Schrödinger equation, i.e. $-\Psi_{xx} + u\Psi = k^2\Psi$. Putting

$$\theta = \begin{bmatrix} \Psi_x \\ \Psi \end{bmatrix},$$

we can express the Schrödinger equation as $\theta_x = X\theta$, where

$$X = \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix},$$

with $\lambda := k^2$. Now we are interested in finding an operator T such that $X_t - T_x + XT - TX = 0$ yields the KdV equation. We look for T in the following form:

$$T = \begin{bmatrix} a_2\lambda^2 + a_1\lambda + a_0 & b_2\lambda^2 + b_1\lambda + b_0 \\ c_2\lambda^2 + c_1\lambda + c_0 & d_2\lambda^2 + d_1\lambda + d_0 \end{bmatrix},$$

where the coefficients a_i, b_i, c_i, d_i for $i = 0, 1, 2$ depend on x, t but do not depend on λ. Imposing (1.0.6)and using a prime to denote the x-derivative, we get

$$\begin{bmatrix} 0 & u_t \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a_2^{'}\lambda^2 + a'_1\lambda + a'_0 & b'_2\lambda^2 + b'_1\lambda + b'_0 \\ c'_2\lambda^2 + c'_1\lambda + c'_0 & d'_2\lambda^2 + d'_1\lambda + d'_0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2\lambda^2 + a_1\lambda + a_0 & b_2\lambda^2 + b_1\lambda + b_0 \\ c_2\lambda^2 + c_1\lambda + c_0 & d_2\lambda^2 + d_1\lambda + d_0 \end{bmatrix}$$
$$- \begin{bmatrix} a_2\lambda^2 + a_1\lambda + a_0 & b_2\lambda^2 + b_1\lambda + b_0 \\ c_2\lambda^2 + c_1\lambda + c_0 & d_2\lambda^2 + d_1\lambda + d_0 \end{bmatrix} \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choosing the coefficients a_i, b_i, c_i, d_i for $i = 0, 1, 2$ in such a way that

$$T = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix},$$

we can write the compatibility condition (1.0.6) as

$$\begin{bmatrix} 0 & u_t \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -u_{xx} & 2\lambda u_x + 4uu_x - u_{xxx} \\ 2u_x & -u_{xx} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}$$
$$- \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix} \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, after straigtforward calculations and displaying the $(1,1)$, $(1,2)$, and $(2,2)$ entries in the previous matrix equation, we get

$$\begin{bmatrix} \begin{matrix} -u_{xx} + (u-\lambda)(4\lambda + 2u) \\ +4\lambda^2 - 2\lambda u - 2u^2 + u_{xx} \end{matrix} & \begin{matrix} u_t - 2\lambda u_x \\ -4uu_x + u_{xxx} - 2(u-\lambda)u_x \end{matrix} \\ \\ 2u_x + u_x + u_x & \begin{matrix} u_{xx} - (u-\lambda)(4\lambda + 2u) \\ -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \end{matrix} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ \\ 0 & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} 0 & u_t - 6uu_x + u_{xxx} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the compatibility condition (1.0.6) holds if the KdV equation is satisfied.

- Let us consider the first-order system in the following form, i.e.

$$\frac{d}{dx}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -i\lambda & \frac{1}{2}u_x \\ -\frac{1}{2}u_x & i\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Let us choose X and T as

$$X = \begin{bmatrix} -i\lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & i\lambda \end{bmatrix}, \quad T = \frac{i}{4\lambda}\begin{bmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{bmatrix},$$

in such a way that the spatial evolution is given by $\theta_x = X\theta$ and the temporal evolution is described by $\theta_t = T\theta$, with

$$\theta = \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

It is possible to prove that the compatibility condition (1.0.6) holds if the *sine-Gordon* equation, $u_{xt} = \sin u$ is satisfied.

- The integrability of the *Dym equation* $u_t = u^3 u_{xxx}$ can be established also with the AKNS method. It is sufficient make the following choices for the AKNS pair:

$$X = \begin{bmatrix} 0 & \frac{\lambda}{u^2} \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 2\lambda u_x & 4\frac{\lambda^2}{u} - 2\lambda u_{xx} \\ 4\lambda u & -2\lambda u_x \end{bmatrix},$$

in such a way that the spatial and temporal evolutions are described, respectively, by $\theta_x = X\theta$ and $\theta_t = T\theta$, where

$$\theta = \begin{bmatrix} \Psi_x \\ \Psi \end{bmatrix}.$$

If one tries to impose the compatibility condition $X_t - T_x + Xt - TX = 0$, one finds the Dym equation and this proves its integrability.

- Let us consider the Degasperis-Procesi equation

$$u_t - u_{xxt} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

where κ is a constant. Putting

$$m := u - u_{xx}, \quad \theta = \begin{bmatrix} \Psi_{xx} \\ \Psi_x \\ \Psi \end{bmatrix},$$

and

$$X = \begin{bmatrix} 0 & 1 & (m + \frac{2}{3}k)\lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} \frac{1}{\lambda} - u_x & \frac{2}{3}k & u_x - \lambda u(m + \frac{2}{3}k) \\ -u & \frac{1}{\lambda} & u + \frac{2}{3}k \\ \frac{1}{\lambda} & -u & u_x \end{bmatrix},$$

we know the spatial and temporal evolutions $\theta_x = X\theta$ and $\theta_t = T\theta$. Direct calculations show that if one imposes the compatibility condition $X_t - T_x + Xt - TX = 0$, the Degasperis-Procesi equation is obtained, which implies its integrability.

- Let us now establish the integrability to the KPI equation:

$$(u_t - 6uu_x + u_{xxx})_x = -3u_{yy},$$

where u is a function depending on x, y, t. We have already observed that this equation can be written as

$$u_t - 6uu_x + u_{xxx} = -3\partial_x^{-1} u_{yy}.$$

In order to verify its integrability by using the AKNS method we use the AKNS pair X and T as

$$X = -\frac{1}{i}\partial_x^2 + \frac{u - \lambda}{i}, \quad T = -4\partial_x^3 + 6u\partial_x + \partial_x - 3i\partial_x^{-1} u_y.$$

If we impose the compatibility condition (1.0.6) we get the KPI equation.

- Let us analyze the NLS equation $iu_t + u_{xx} + 2|u|^2 u = 0$. It is well known that the LODE corresponding to this NPDE is the Zakharov-Shabat system

$$\frac{d}{dx}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -i\lambda & u \\ -u^* & i\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Writing this system in the form $\theta_x = X\theta$ where we have defined

$$X = \begin{bmatrix} -i\lambda & u(x,t) \\ -u^*(x,t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator T as

$$T = \begin{bmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & 2i\lambda^2 - i|u|^2 \end{bmatrix},$$

and the compatibility condition (1.0.6) yields the NLS equation.

Alternative Lax method method

Finally, I describe an *Alternative Lax method* which allows to derive an integrable NPDE corresponding to a given LODE. We can illustrate this method as follows: Suppose we write the LODE as

$$L\Psi = \lambda\Psi,$$

where the linear differential operator contains the highest x-derivative ∂_x^n for some n, and suppose we know the time evolution of the solution of the previous equation is described by

$$\Psi_t = B\Psi,$$

where B is usually a linear differential operator in the spatial coordinate x. The integrable NPDE corresponding to the LODE is obtained by imposing the compatibility condition

$$\partial_t \partial_x^n \Psi = \partial_x^n \partial_t \Psi, \tag{1.0.7}$$

where the left side is obtained by taking the time derivative of $L\Psi = \lambda\Psi$ and right side is obtained by taking the n-th derivative in x of $\Psi_t = B\Psi$. I explain the method by considering several examples.

Examples.

- The KdV equation is obtained from

$$(-\frac{d^2}{dx^2} + u(x,t))\Psi = \lambda\Psi, \qquad \Psi_t = (4\lambda + 2u)\Psi_x - u_x\Psi$$

by imposing the compatibility condition $\partial_t \partial_x^2 \Psi = \partial_x^2 \partial_t \Psi$.
In fact, writing the spatial and temporal evolutions given above in the form

$$\Psi_{xx} = (u-\lambda)\Psi,$$
$$\Psi_t = (4\lambda + 2u)\Psi_x - u_x\Psi,$$

we get

$$\begin{aligned}&\Psi_{xxt} = u_t\Psi + (u-\lambda)\Psi_t,\\ &\Psi_{txx} = u_{xx}\Psi_x + u_x\Psi_{xx} + (4u_x u - 2\lambda u_x)\Psi + (2u^2 - 2\lambda u - 4\lambda^2)\Psi_x\\ &- u_{xxx}\Psi - u_{xx}\Psi_x,\end{aligned}$$

where we need to express the right hand sides in terms of Ψ and Ψ_x alone. The compatibility condition $\Psi_{txx} = \Psi_{xxt}$ implies that

$$\begin{aligned}&[u_t - u_x(u-\lambda)]\,\Psi + (2\lambda u + 2u^2 - 4\lambda^2)\Psi_x =\\ &[u_x(u-\lambda) + 4u_x u - 2\lambda u_x - u_{xxx}]\,\Psi + (u_{xx} + 2\lambda u + 2u^2 - 4\lambda^2 - u_{xx})\Psi_x.\end{aligned}$$

The coefficients of Ψ and Ψ_x, respectively, on both sides should match, yielding

$$u_t - 6uu_x + u_{xxx} = 0,$$

i.e. the Kdv equation holds.

- The sine-Gordon equation

$$u_{xt} = \sin u,$$

can be obtained as the compatibility condition

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_{xt} = \begin{bmatrix}\xi\\ \eta\end{bmatrix}_{tx},$$

from the system

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_x = \begin{bmatrix}-i\lambda & -\frac{u_x}{2}\\ \frac{u_x}{2} & i\lambda\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix},$$
$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_t = \frac{i}{4\lambda}\begin{bmatrix}\cos u & \sin u\\ \sin u & \cos u\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}.$$

- The NLS equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0,$$

can be obtained as the compatibility condition

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_{xt} = \begin{bmatrix}\xi\\ \eta\end{bmatrix}_{tx},$$

from the system

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_x = \begin{bmatrix}-i\lambda & u\\ \mp u^* & i\lambda\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix},$$
$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_t = \begin{bmatrix}-2i\lambda^2 \pm i|u|^2 & 2\lambda u + iu_x\\ \mp 2\lambda u^* \pm iu_x^* & 2i\lambda^2 \mp i|u|^2\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}.$$

- The Dym equation

$$u_t = u^3 u_{xxx},$$

can be obtained as the compatibility condition $\Psi_{xxt} = \Psi_{txx}$ from the system

$$\begin{cases} \Psi_{xx} = \frac{\lambda}{u(x,t)^2}\Psi \\ \Psi_t = 4\lambda u(x,t)\Psi_x - 2\lambda u_x(x,t)\Psi. \end{cases}$$

- The Degasperis-Procesi equation

$$u_t - u_{xxt} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

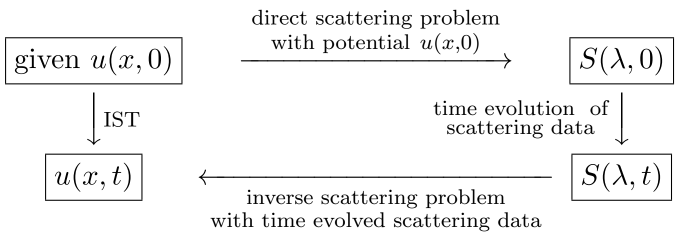
where κ is a constant, can be obtained as the compatibility condition $\Psi_{xxxt} = \Psi_{txxx}$ from the system

$$\begin{cases} \Psi_{xxx} = \Psi_x + \lambda(m(x,t) + \frac{2}{3}\kappa)\Psi \\ \Psi_t = \frac{1}{\lambda}\Psi_{xx} - u(x,t)\Psi_x + u_x(x,t)\Psi, \end{cases}$$

where we have defined $m(x,t) := u(x,t) - u_{xx}(x,t)$.

Lecture 7 and 8

Now I would like to describe the *Hamiltonian formulation of IST*. In order to do that, let us consider the diagram which shows how the IST works:



In classical mechanics is well-known that the problem of motion can be attached using either the Lagrangian or the Hamiltonian formalism. Let us recall the basic facts about the Hamiltonian language. If q_h $(h = 1, \dots, N)$ denotes the Lagrangian variable which characterize the observed system, it is possible to define the so-called *momentum* p_h in terms of the Lagrangian function $\mathcal{L}(q_h, \dot{q}_h, t)$ as

$$p_h = \frac{\partial \mathcal{L}}{\partial \dot{q}_h}.$$

Then we can introduce the Hamiltonian function $H(p, q, t)$ of the variables p_h, q_h, t in such a way that

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p}. \end{aligned}$$

Let us consider some illustrative examples:

- Suppose that $H = \frac{p^2}{2m} + V(q)$. We easily get

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}, \end{aligned}$$

  where $-\frac{\partial V}{\partial q} = \boldsymbol{F}$ represents the force. As a result we find the equation of motion $m\ddot{q} = \dot{p} = \boldsymbol{F}$;

- Suppose that $H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(q_1, q_2)$. We obtain

$$\begin{aligned} \dot{q_1} &= -\frac{\partial H}{\partial p_1} = \frac{p_1}{m_1} \\ \dot{q_2} &= -\frac{\partial H}{\partial p_2} = \frac{p_1}{m_1} \\ \dot{p_1} &= -\frac{\partial H}{\partial q_1} = -\frac{\partial V}{\partial q_1} \\ \dot{p_2} &= -\frac{\partial H}{\partial q_2} = -\frac{\partial V}{\partial q_2}. \end{aligned}$$

  The above equations form a coupled first order in t equations describing the time evolution of $(p, q) := (p_1, p_2, q_1, q_2)$.

It is interesting to observe that the *Hamilton's principle of lenght actions* is valid. The Hamilton principle of lenght actions requires $\int_{t_1}^{t_2} H\,dt$ must be smallest. The path $q(t)$ that gives you the smallest action is the path to use. Then find the evolution from t_1 to t_2 of the Hamiltonian system is equivalent to determine $q(t)$ and $p(t)$.
Now we can approach the problem connected with the IST: Given an integrable NPDE we can identify the variable q as the unknown function $u(x,t)$ which appear in the NPDE. Furthermore, we have some freedom in the choice of the momentum variables: one possibility is to take $p = u_x(x,t)$. Since the Hamilton's equation $q_t = \frac{\partial H}{\partial P}$ holds, the derivative with respect to time t (of the variable q) allows us to identify the Hamiltonian H in terms of the NPDE. In particular, we find that the Hamiltonian function can be expressed in terms of a *density function* h in the following way:

$$H(p,q,t) = \int_{-\infty}^{+\infty} dx\, h(q, q_x, q_{xx}, \ldots) \quad .$$

Then we have

$$\delta H = \int_{-\infty}^{\infty} dx \left(\frac{\partial h}{\partial q}\delta q + \frac{\partial h}{\partial q_x}\delta q_x + \frac{\partial h}{\partial q_{xx}}\delta q_{xx} + \ldots \right),$$

taking into account that

$$\begin{aligned}
\frac{\partial h}{\partial q_x}\delta q_x &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial q_x}\delta q\right) - \left(\frac{\partial}{\partial x}\frac{\partial h}{\partial q_x}\right)\delta q \\
\frac{\partial h}{\partial q_{xx}}\delta q_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial q_{xx}}\delta q_x\right) - \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{xx}}\delta q\right) - \left(\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{xx}}\right)\delta q\right]
\end{aligned}$$

and integrating by parts, we get

$$\begin{aligned}
\frac{\delta H}{\delta q} &= -\left[\frac{\partial h}{\partial q} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial q_x}\right) + \frac{\partial^2}{\partial x^2}\frac{\partial h}{\partial q_{xx}} + \ldots\right], \\
\frac{\delta H}{\delta p} &= \frac{\partial h}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial p_x}\right) + \frac{\partial^2}{\partial x^2}\frac{\partial h}{\partial p_{xx}} + \ldots,
\end{aligned}$$

which show as the Hamiltonian equations can be expressed in terms of the density function $h(p,q,t)$.
In 1971 Faddeev and Zakharov proposed the Hamiltonian formulation of the KdV equation $u_t - 6uu_x + u_{xxx} = 0$. They chosen the variable q as $q = u(x,t)$ (where u is the unknown function appearing in the KdV equation) and they wrote this equation in the following form

$$\dot{q} = \frac{\partial}{\partial x}\left(3u^2 - u_{xx}\right).$$

Then they defined the Hamilton function $H = \int_{-\infty}^{\infty} dx\, h(u, u_x, \ldots)$ depending only on q_h, and did not mention the momentum variable p. Successively, Ablowitz and Clarkson revisited the same equation introducing the momentum variable $p = u_x$ and writing the $H = \int_{-\infty}^{+\infty} dx\, h(p, q, p_x, q_x, p_{xx}, q_{xx} \cdots)$. Consequently, they got the Hamilton equations as follows:

$$\begin{aligned}
\dot{p} &= -\left[\frac{\partial h}{\partial q} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial q_x}\right) + \frac{\partial^2}{\partial x^2}\frac{\partial h}{\partial q_{xx}} + \ldots\right], \\
\dot{q} &= \frac{\partial h}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial p_x}\right) + \frac{\partial^2}{\partial x^2}\frac{\partial h}{\partial p_{xx}} + \ldots .
\end{aligned}$$

On the other hand, in the same period, Faddeev-Takhtajan developed the Hamiltonian formalism for the NLS equation in the scalar case, i.e. for the equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0,$$

where i represents the imaginary unit, the sign $+$ in the third terms on the right side corresponds to the defocussing case while the sign $-$ corresponds to the focussing case. Indicating by u^* the complex conjugate of u, they considered the equation $-iu_t^* + u_{xx}^* \pm 2|u|^2 u = 0$, the variables p and q and the Hamiltonian function H as

$$q = u, \quad p = u^*, \quad H = \int_{-\infty}^{+\infty} dx \left(q_x p_x \mp q^2 (p)^2\right). \tag{1.0.8}$$

As a consequence the density function h can be defined as

$$h(p, q, p_x, q_x) = q_x p_x \mp q^2 (p)^2, \tag{1.0.9}$$

and the Hamilton's equations become

$$\begin{aligned} q_t &= \frac{\partial h}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial p_x}\right) \\ p_t &= -\left[\frac{\partial h}{\partial q} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial q_x}\right)\right]. \end{aligned}$$

In fact, starting from the first of the previous equations and computing the derivative of the density function given by (1.0.9), we easily get

$$q_t = -i\left[\mp 2pq^2 - \frac{\partial}{\partial x}(q_x)\right],$$

can be written in the equivalent form $iu_t + u_{xx} \pm 2|u|^2 u = 0$ via the definitions (1.0.8). In a general context we can approach at the Hamilton formulation as follows: Given an integrable NPDE we have to identify the Hamilton function H and the density function h in terms of $u, u_x, u_{xx}, \ldots$ so that if one is able to solve the Hamilton equation

$$\begin{aligned} q_t &= \frac{\delta H}{\delta p} \\ p_t &= -\frac{\delta H}{\delta q}, \end{aligned} \tag{1.0.10}$$

obtain the solution of NPDE because $q = u(x,t)$ and $p = u_x(x,t)$. Here we remark the arbitrary in the choice of the momentum variables, although the second of the previous equation represents the more common choices. In order determine the solutions of (1.0.10), it can be convenient change the coordinate and pass from (p, q) to (P, Q) in such a way that the Hamilton's function H can be expressed in terms of the new variables $H(p(P,Q), q(P,Q)) = H(P,Q)$ and the Hamilton's equation becomes

$$\begin{aligned} Q_t &= \frac{\delta H}{\delta P} = c \\ P_t &= -\frac{\delta H}{\delta Q} = 0, \end{aligned} \tag{1.0.11}$$

where c is a constant. The variables (P, Q) which allow to write the Hamilton's equation in the particular form (1.0.12) are called *canonical coordinate.* In particular, the variable P is called *action* and the variable Q *angle.* From equations (1.0.12) we immediately get

$$Q = ct + a \qquad (1.0.12)$$
$$P = b,$$

where a, b are constants. The canonical coordinates satisfies the following equations

$$\{p_n, q_m\} = \delta_{n,m}, \quad \{p_n, p_m\} = 0, \quad \{q_n, q_m\} = 0,$$

where the brackets denotes the Poisson brackets $\{f, g\} = \sum_{i=1}^{\infty}\left(\frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n}\right)$ and $\delta_{n,m}$ represents the Kronecher's delta.
Now using the inverse of the transformation which permits to pass from (p, q) to (P, Q), we obtain the variables (p, q) and knowing $q(t)$ we succeed in reconstructing the solution to the integrable NPDE.
The relationship between the Hamiltonian formalism and the Inverse Scattering Transform can be made clear in the following sense: the variables P and Q are related to the scattering data $S(\lambda, t)$ (usually Q correspond to the $\arg R(\lambda, t)$ and P to $|T(\lambda, t)|^2$) instead p, q are connected to the unknown function $u(x, t)$. In other words, the passage from (P, Q) to (p, q) is equivalent to handle the Inverse Scattering Problem with time evolved scattering data. Obviously, the mathematical difficulties consists of the discovery of the transformation which allows us to pass from the variable (p, q) to the canonical variables (P, Q).
Let us illustrate the passage to canonical variable in an important case: the KdV equation. As we have already mentioned the LODE corresponding to the KdV is the Schrödinger equation. The Jost solution of this equation are defined in the following way:

$$f_l(k, x, t) \sim e^{ikx}, \quad \text{as } x \to +\infty$$
$$f_r(k, x, t) \sim e^{-ikx}, \quad \text{as } x \to -\infty,$$

where $f_l(k, x, t)$ is the so-called *left Jost solution* and $f_r(k, x, t)$ is the *right Jost solution.* It is possible to prove that the expression of f_l is given by

$$f_l(k, x, t) = e^{ikx} + \int_x^{+\infty} dy\, \frac{\sin k(y - x)}{k} u(y, t) f_l(k, y, t),$$

and similar expression can be obtained for $f_r(k, x, t)$. Now, we can compute the reflection coefficient on the left $L(k, t)$ and the transmission coefficient $T(k, t)$ from the asymptotic expansion

$$\frac{e^{ikx}}{T(k, t)} + \frac{e^{-ikx} L(k, t)}{T(k, t)} = f_l(k, -\infty, t) = e^{ikx} + \int_{-\infty}^{+\infty} dy\, \frac{\sin k(y - x)}{k} u(y, t) f_l(k, y, t)\,.$$

In fact, from the last equation we get

$$\frac{1}{T(k, t)} = 1 + \int_{-\infty}^{+\infty} dy\, \frac{e^{-iky}}{2ik} u(y, t) f_l(k, y, t)$$
$$\frac{L(k, t)}{T(k, t)} = -\int_{-\infty}^{+\infty} dy\, \frac{e^{iky}}{2ik} u(y, t) f_l(k, y, t),$$

which allow to write

$$\begin{aligned} T(\lambda,t) &= 1+\int_{-\infty}^{+\infty} dy\, u(y,t) \\ \frac{L(k,t)}{T(k,t)} &= -\int_{-\infty}^{+\infty} dy\, u(y,t). \end{aligned}$$

If now one expand $T(\lambda,t)$ in powers of $\frac{1}{\lambda}$ obtain

$$\ln\frac{1}{|T(\lambda,t)|} = \frac{1}{\lambda}(\ldots)+\frac{1}{\lambda^2}(\ldots)+\frac{1}{\lambda^3}(\ldots),$$

where the terms into the $(\ldots)$ are, respectively, $\int_{-\infty}^{+\infty} dyu(y,t)$, $\int_{-\infty}^{+\infty} dyu_y(y,t)$, $\int_{-\infty}^{+\infty} dyu_{yy}(y,t)$, etc.

Toda Lattice Equation. In order to understand better the Toda lattice equation, let us recall the model analyzed by Fermi, Pasta and Ulam. This one dimensional model consists of 64 particles of mass m joint through springs each one having the same elastic constant k. Furthermore, in this model the requirement is that the elastic force does not obey at the Hook's law but acts in a nonlinear way in such a way that

$$F=-kx+\epsilon^2x^2,\quad V=\frac{1}{2}kx^2-\frac{\epsilon}{3}x^3,$$

where V represents the potential. Now, let us suppose that the number of particles are infinite and the potential of the system is given by

$$V(r)=e^{-r}+r-1=\sum_{i=2}^{+\infty}(-1)^i\frac{r^i}{i!}.$$

Under these assumptions we can introduce the variables q_n- representing the displacement of the particle sitting at the n-th lattice point- and the momentum variable $p_n=\dot{q_n}$, where $n\in\mathbb{Z}$. We also define the Hamilton's function

$$H=\sum_{n=-\infty}^{\infty}\left(\frac{p_n^2}{2}+V(q_{n+1}-q_n)\right),$$

so that the Hamilton's equations give us

$$\begin{aligned} \dot{q}_n &= \frac{\partial H}{\partial p_n}=p_n \\ \dot{p}_n &= -\frac{\partial H}{\partial q_n}=V'(q_{n+1}-q_n)-V'(q_n-q_{n-1})=e^{-(q_n-q_{n-1})}-e^{-(q_{n+1}-q_n)}, \end{aligned}$$

where we have taken into account that $V'(r)=-e^{-r}+1$. Then we immediately obtain

$$\ddot{q}_n=\dot{p}_n=e^{-(q_n-q_{n-1})}-e^{-(q_{n+1}-q_n)} \tag{1.0.13}$$

which is called *discrete, nonlinear Toda lattice equation.* We can easily verify that the coordinates (p_n,q_n) satisfies the following equations

$$\{p_n,q_m\}=\delta_{n,m},\quad \{p_n,p_m\}=0,\quad \{q_n,q_m\}=0\,. \tag{1.0.14}$$

In fact, recalling the definition of the Poisson brackets $\{f,g\} = \sum_{i=1}^{\infty}\left(\frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n}\right)$ we calculate

$$\{p_n, p_m\} = \sum_{j=-\infty}^{\infty}\left[\frac{\partial p_n}{\partial q_j}\frac{\partial p_m}{\partial p_j} - \frac{\partial p_n}{\partial p_j}\frac{\partial p_m}{\partial q_j}\right] = 0 \tag{1.0.15}$$

$$\{p_n, q_m\} = \sum_{j=-\infty}^{\infty}\left[\frac{\partial p_n}{\partial q_j}\frac{\partial q_m}{\partial p_j} - \frac{\partial p_n}{\partial p_j}\frac{\partial q_m}{\partial q_j}\right] = \sum_{j=-\infty}^{\infty}[\delta_{n,j}\delta_{m,j}] = \delta_{n,m}$$

$$\{q_n, q_m\} = \sum_{j=-\infty}^{\infty}\left[\frac{\partial q_n}{\partial q_j}\frac{\partial q_m}{\partial p_j} - \frac{\partial q_n}{\partial p_j}\frac{\partial q_m}{\partial q_j}\right] = 0\,.$$

where we have used $\frac{\partial p_n}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial \dot{q}_n}{\partial q_j}\right) = \frac{d}{dt}\delta_{m,j} = 0$ and, analogously, we can prove the following identities (also used in order to establish the validity of (1.0.15)) $\frac{\partial p_m}{\partial q_j} = 0$, $\frac{\partial q_n}{\partial p_j} = 0$, $\frac{\partial q_m}{\partial p_j} = 0$. Then, for solving (1.0.13) we have to determine the canonical coordinates J_n, θ_n (called, respectively, action and angle) such that the equations (1.0.14) hold and the following ones have to be satisfied

$$\dot{J}_n = 0$$
$$\dot{\theta}_n = c,$$

where c is a constant.
Finally, we prove that the Toda lattice equation is integrable and then can be solved via IST. To prove the integrability we build the Lax pair of the equation (1.0.13), i.e. we are looking for two operators L_n and A_n satisfying

$$L_n\Psi_n = \lambda\Psi_n$$
$$\dot{\Psi}_n = A_n\Psi_n$$

and such that

$$\dot{L}_n = L_nA_n - A_nL_n. \tag{1.0.16}$$

In order to develop the calculations, we introduce the operators S^+ and S^- as

$$\left(S^+f\right)(n) = f(n+1), \quad \left(S^-f\right)(n) = f(n-1),$$

the coefficients $a_n = \frac{1}{2}e^{-\frac{1}{2}(q_{n+1}-q_n)}$ and $b_n = -\frac{1}{2}\dot{q}_n$, and the operators $L_n = a_nS^+ + \tilde{a}_{n-1}S^- + b_n$, and $A_n = c_nS^+ - \tilde{c}_{n-1}S^-$. Since we want impose equation (1.0.16), we start computing $\dot{L}_n$, L_nA_n and A_nL_n

$$\begin{aligned}
&\dot{L}_n = \dot{a}_nS^+ + \dot{\tilde{a}}_{n-1}S^- + \dot{b}_n\,,\\
&L_nA_n = \left(a_nS^+ + \tilde{a}_{n-1}S^- + b_n\right)\left(c_nS^+ - \tilde{c}_{n-1}S^-\right) =\\
&a_nc_{n+1}S^{++} - \tilde{a}_{n-1}\tilde{c}_{n-2}S^{--} + b_nc_nS^+ - b_n\tilde{c}_{n-1}S^- - a_n\tilde{c}_n + \tilde{a}_{n-1}c_{n-1}\,,\\
&A_nL_n = \left(c_nS^+ - \tilde{c}_{n-1}S^-\right)\left(a_nS^+ + \tilde{a}_{n-1}S^- + b_n\right) =\\
&c_na_{n+1}S^{++} - \tilde{c}_{n-1}\tilde{a}_{n-2}S^{--} + c_nb_{n+1}S^+ - \tilde{c}_{n-1}b_{n-1}S^- + c_n\tilde{a}_n - \tilde{c}_{n-1}a_{n-1}\,,
\end{aligned}$$

which imply that equation (1.0.16), choosing $\tilde{a}_{n-1} = a_{n-1}$, $c_n = a_n$ and $\tilde{c}_{n-1} = c_{n-1}$, can be written as

$$\begin{aligned}
&\left(a_na_{n+1} - a_na_{n+1}\right)S^{++} + \left(a_{n-2}a_{n-1} - a_{n-1}a_{n-2}\right)S^{--} + \left(\dot{a}_n + b_na_n - a_nb_{n+1}\right)S^+\\
&+ \left(\dot{a}_{n-1} + b_{n-1}a_{n-1} - b_na_{n-1}\right)S^- + \dot{b}_n - a_na_n + a_{n-1}a_{n-1} - a_na_n + a_{n-1}a_{n-1} = 0.
\end{aligned}$$

The coefficients of S^{++} and S^{--} vanish automatically, while the the non-homogeneous coefficient equated to zero is equivalent to the Toda lattice equation $\dot{b}_n - 2a_n^2 + 2a_{n-1}^2 = 0$. In this way we have established the integrability (via the IST) of the discrete version to the KdV equation.