nD Spherical Functions

1 Laplacian in Spherical Coordinates

Let $d = 2, 3, \ldots$. Consider the orthogonal coordinates $u_1, \ldots, u_d$ as well as the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} \ldots & \frac{\partial x_d}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} \ldots & \frac{\partial x_d}{\partial u_2} \\ \frac{\partial x_1}{\partial u_d} & \frac{\partial x_2}{\partial u_d} \ldots & \frac{\partial x_d}{\partial u_d} \end{pmatrix}, \tag{1.1}$$

where $x_1, \ldots, x_d$ are the Cartesian coordinates of $\mathbb{R}^d$. Then the Jacobian matrix has mutually orthogonal rows. Let $h_1, \ldots, h_d$ denote the lengths of the respective row. Then the Jacobian (i.e., the absolute value of the determinant of the Jacobian matrix) equals $h_1 h_2 \ldots h_d$. We also have

$$\nabla_x^2 \psi = \frac{1}{h_1 \ldots h_d} \sum_{i=1}^{d} \frac{\partial}{\partial u_i} \left(\frac{h_1 \ldots h_{i-1} h_{i+1} \ldots h_d}{h_i} \frac{\partial \psi}{\partial u_i} \right). \tag{1.2}$$

Now consider the following orthogonal coordinate transformation:

$$\begin{cases} x_1 = r \cos \varphi_1 \\ x_2 = r \sin \varphi_1 \cos \varphi_2 \\ x_3 = r \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ \vdots \\ x_{d-1} = r \sin \varphi_1 \ldots \sin \varphi_{d-2} \cos \varphi_{d-1} \\ x_d = r \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{d-2} \sin \varphi_{d-1}, \end{cases} \tag{1.3}$$

where $r \geq 0$, $\{\varphi_1, \dots, \varphi_{d-2}\} \subset [0, \pi]$ and $\varphi_{d-1} \in [-\pi, \pi]$. In other words, x_1 and x_d are as above, while

$$x_j = r \prod_{s=1}^{j-1} \sin\varphi_s \cos\varphi_j, \qquad j = 2, 3, \dots, d-2.$$

Then

$$h_r = 1,\ h_{\varphi_1} = r,\ h_{\varphi_2} = r\sin\varphi_1, \dots,\ h_{\varphi_{d-1}} = r\sin\varphi_1 \sin\varphi_2 \dots \sin\varphi_{d-2}. \tag{1.4}$$

so that the Jacobian equals

$$J = r^{d-1} \prod_{s=1}^{d-2} (\sin\varphi_s)^{d-1-s}\,. \tag{1.5}$$

As a result,

$$\begin{aligned}
\nabla^2\psi = &\frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2(\sin\varphi_1)^{d-2}} \frac{\partial}{\partial\varphi_1}\left((\sin\varphi_1)^{d-2}\frac{\partial\psi}{\partial\varphi_1}\right) \\
&+ \frac{1}{r^2(\sin\varphi_1)^2(\sin\varphi_1)^{d-3}} \frac{\partial}{\partial\varphi_2}\left((\sin\varphi_2)^{d-3}\frac{\partial\psi}{\partial\varphi_2}\right) \\
&+ \frac{1}{r^2(\sin\varphi_1\sin\varphi_2)^2(\sin\varphi_3)^{d-4}} \frac{\partial}{\partial\varphi_3}\left((\sin\varphi_3)^{d-4}\frac{\partial\psi}{\partial\varphi_3}\right) \\
&+ \dots + \frac{1}{r^2(\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{d-3})^2\sin\varphi_{d-2}} \frac{\partial}{\partial\varphi_{d-2}}\left(\sin\varphi_{d-2}\frac{\partial\psi}{\partial\varphi_{d-2}}\right) \\
&+ \frac{1}{r^2(\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{d-2})^2} \frac{\partial^2\psi}{\partial\varphi_{d-1}^2}.
\end{aligned} \tag{1.6}$$

In other words,

$$\nabla^2\psi = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r^2}L_B^d\psi, \tag{1.7}$$

where L_B^d is a differential operator in the angular variables called the Beltrami operator on S^{d-1}.

Instead of the above transformation of variables it is often convenient to use the following orthogonal transformation:

$$\begin{cases} x_1 = r\xi_1 \\ x_2 = r\sqrt{1-\xi_1^2}\xi_2 \\ x_3 = r\sqrt{(1-\xi_1^2)(1-\xi^2)}\xi_3 \\ \vdots \\ x_{d-1} = r\sqrt{(1-\xi_1^2)(1-\xi^2)\dots(1-\xi_{d-2}^2)}\cos\varphi_{d-1} \\ x_{d-1} = r\sqrt{(1-\xi_1^2)(1-\xi^2)\dots(1-\xi_{d-2}^2)}\sin\varphi_{d-1}, \end{cases} \tag{1.8}$$

which is related to the previous variables by

$$\xi_j = \cos\varphi_j, \qquad j = 1, \dots, d-2.$$

Thus $\xi_j \in [-1, 1]$ $(j = 1, \dots, d-2)$ and $\varphi_{d-1} \in [-\pi, \pi]$. Then $h_r = 1$ and $h_{\varphi_{d-1}}$ are as before (modulo $\xi_j = \cos\varphi_j$ for $j = 1, \dots, d-2$), but modulo this change we have $h_{\xi_j} = h_{\varphi_j}/(1-\xi_j^2)^{1/2}$ for $j = 1, \dots, d-2$. We then find (1.7), where

$$(L_B^d\psi)(\xi_1, \xi_2, \dots, \xi_{d-2}, \varphi_{d-1}) = -\frac{1}{(1-\xi_1^2)^{\frac{d-3}{2}}}\frac{\partial}{\partial\xi_1}\left((1-\xi_1^2)^{\frac{d-1}{2}}\frac{\partial\psi}{\partial\xi_1}\right) + \frac{1}{1-\xi_1^2}(L_B^{d-1}\psi)(\xi_2, \dots, \xi_{d-2}, \varphi_{d-1}). \tag{1.9}$$

The first few Beltrami operators are as follows:

$$(L_B^2\psi)(\phi_1) = -\frac{\partial^2\psi}{\partial\phi_1^2}, \tag{1.10}$$

$$(L_B^3\psi)(\phi_1) = -\left(\frac{\partial}{\partial\xi_1}(1-\xi_1^2)\frac{\partial\psi}{\partial\xi_1}\right) - \frac{\partial^2\psi}{\partial\phi_2^2}, \tag{1.11}$$

$$(L_B^4\psi)(\phi_1) = -\frac{1}{\sqrt{1-\xi_1^2}}\left(\frac{\partial}{\partial\xi_1}(1-\xi_1^2)^{3/2}\frac{\partial\psi}{\partial\xi_1}\right) - \frac{1}{1-\xi_1^2}\left[\left(\frac{\partial}{\partial\xi_2}(1-\xi_2^2)\frac{\partial\psi}{\partial\xi_2}\right) + \frac{\partial^2\psi}{\partial\phi_3^2}\right]. \tag{1.12}$$

In order to separate the variables in the d-D Schrödinger equation with spherically symmetric potential

$$\nabla^2\psi + [k^2 - V(r)]\psi = 0, \tag{1.13}$$

we write

$$\psi(r,\xi_1,\ldots,\xi_{d-2},\varphi_{d-1}) = R(r)\Psi(\xi_1,\ldots,\xi_{d-2},\varphi_{d-1})$$

and derive the ordinary differential equation

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}R'(r)\right) + \left(k^2 - V(r) - \frac{\text{const.}}{r^2}\right)R(r) = 0 \tag{1.14}$$

as well as the partial differential equation

$$L_B^d\Psi = \text{const.}\Psi. \tag{1.15}$$

To find the common constant, we recall that the eigenfunctions of the Beltrami operator are exactly the d-variate harmonic polynomials of degree $n_1 = 0,1,2,\ldots$ restricted to S^{d-1}. Substituting

$$\psi(r,\theta) = r^{n_1}\,y(\theta), \qquad \theta \in S^{d-1},$$

into the Laplace equation and noting that

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}r^{n_1}\right) = n_1(n_1+d-2)r^{n_1-2},$$

we obtain

$$L_B^d\psi = n_1(n_1+d-2)\psi, \qquad n_1 = 0,1,2,\ldots. \tag{1.16}$$

Hence (1.14) reduces to the ordinary differential equation

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}R'(r)\right) + \left(k^2 - V(r) - \frac{n_1(n_1+d-2)}{r^2}\right)R(r) = 0. \tag{1.17}$$

Thus if $V(r) \equiv 0$, its only solutions finite as $r \to 0^+$ are the functions

$$R(r) \simeq \frac{J_{n_1+\frac{1}{2}(d-2)}(kr)}{(kr)^{\frac{1}{2}(d-2)}}. \tag{1.18}$$

Next let

$$\psi(\xi_1,\xi_2,\ldots,\xi_{d-2},\phi_{d-1}) = \tilde{\phi}(\xi_1)\Phi(\xi_2,\ldots,\xi_{d-2},\phi_{d-1}). \tag{1.19}$$

Using (1.19) in (1.9) we get

$$
\begin{aligned}
n_1(n_1+d-2) = &-\frac{1}{(1-\xi_1^2)^{\frac{d-3}{2}}\tilde{\phi}(\xi_1)}\frac{d}{d\xi}\left((1-\xi_1^2)^{\frac{d-1}{2}}\tilde{\phi}'(\xi_1)\right) \\
&+\frac{1}{1-\xi_1^2}\,n_2(n_2+d-3),
\end{aligned}
$$

where n_1 and n_2 are integers satisfying $0 \le n_2 \le n_1$. Substituting

$$\tilde{\phi}(\xi_1) = (1-\xi_1^2)^{\frac{1}{2}n_2}\phi(\xi_1)$$

we get after some calculations

$$(1-\xi_1^2)\phi''(\xi_1)-(2n_2+d-1)\xi_1\phi'(\xi_1)+(n_1-n_2)(n_1+n_2+d-2)\phi(\xi_1)=0. \quad (1.20)$$

Substituting $\phi(\xi_1) = \sum_{s=0}^{\infty} c_s\xi_1^s$ into (1.20) it is easily seen that (1.20) has a polynomial solution of degree n_1-n_2. Also, (1.20) can be written in the form

$$\frac{d}{d\xi_1}\left((1-\xi_1^2)^{n_2+\frac{d-1}{2}}\phi(\xi_1)\right) = (n_1-n_2)(n_1+n_2+d-2)(1-\xi_1^2)^{n_2+\frac{d-3}{2}}\phi(\xi_1). \quad (1.21)$$

For $\alpha > -1$ and $s = 0, 1, 2, \ldots$ we now denote by $p_s^{(\alpha)}(\xi)$ the real polynomials with positive leading coefficient such that

$$\int_{-1}^{1} p_l^{(\alpha)}(\xi)p_r^{(\alpha)}(\xi)(1-\xi^2)^{\alpha}\,d\xi = \delta_{l,r}.$$

Then

$$\phi(\xi_1) \simeq p_{n_1-n_2}^{(n_2+\frac{d-3}{2})}(\xi_1). \quad (1.22)$$

Note that these polynomials are proportional to the Jacobi polynomials $P_{n_1-n_2}^{(n_2+\frac{d-3}{2})}(\xi_1)$.

We now repeat the above procedure $d-2$ times and observe that the functions of φ_{d-1} are the trigonometric functions 1 for $n_{d-1}=0$ and $\cos(n_{d-1}\varphi_{d-1})$ and $\sin(n_{d-1}\varphi_{d-1})$ for $n_{d-1} = 1, 2, 3, \ldots$. Thus corresponding to the eigenvalue $n_1(n_1+d-2)$ of the Beltrami operator L_B^d on $L^2(S^{d-1})$ we get the following normalized eigenfunctions:

$$\begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\prod_{s=1}^{d-2} (1-\xi_s^2)^{\frac{1}{2}n_{s+1}} p_{n_s-n_{s+1}}^{(n_{s+1}+\frac{1}{2}(d-2-s))}(\xi_s), & n_1 \geq \ldots \geq n_{d-1} = 0, \\ \dfrac{1}{\sqrt{\pi}} \displaystyle\prod_{s=1}^{d-2} (1-\xi_s^2)^{\frac{1}{2}n_{s+1}} p_{n_s-n_{s+1}}^{(n_{s+1}+\frac{1}{2}(d-2-s))}(\xi_s) \cos(n_{d-1}\varphi_{d-1}), & n_1 \geq \ldots \geq n_{d-1} \geq 1, \\ \dfrac{1}{\sqrt{\pi}} \displaystyle\prod_{s=1}^{d-2} (1-\xi_s^2)^{\frac{1}{2}n_{s+1}} p_{n_s-n_{s+1}}^{(n_{s+1}+\frac{1}{2}(d-2-s))}(\xi_s) \sin(n_{d-1}\varphi_{d-1}), & n_1 \geq \ldots \geq n_{d-1} \geq 1. \end{cases} \tag{1.23}$$

Let us now examine the degeneracy of the eigenvalues of L_B^d. The number of linearly independent solutions of the eigenvalue equation

$$L_B^d \psi = n_1(n_1 + d - 2)\psi$$

equals

$$\begin{aligned} N_{n_1}^d &= \#\{(n_1, n_2, \ldots, n_{d-2}, 0) \in \mathbb{Z}^{d-1} : n_1 \geq n_2 \geq \ldots \geq n_{d-2} \geq 0\} \\ &\quad + 2\#\{(n_1, n_2, \ldots, n_{d-1}) \in \mathbb{Z}^{d-1} : n_1 \geq n_2 \geq \ldots \geq n_{d-2} \geq n_{d-1} \geq 1\} \\ &= N_{n_1}^{d-1} + 2\sum_{s=1}^{n_1} N_{n_1-s}^{d-1} = \sum_{s=0}^{n_1} (2 - \delta_{s,0}) N_{n_1-s}^{d-1}, \end{aligned} \tag{1.24}$$

where $N_{n_1}^2 = 2\delta_{n_1,0}$, $N_{n_1}^3 = 2n_1 + 1$ and $N_{n_1}^4 = 2n_1(n_1+1) + 1$. By induction we easily prove that in general

$$N_{n_1}^d = \frac{(2n_1 + d - 2)(n_1 + d - 3)!}{(n_1)!(d-2)!}. \tag{1.25}$$