Cornelis VAN DER MEE, Spring 2008, Math 3330, Sample Exam 2

Name:.............................. **Grade**:.......... **Rank**:...........

To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Consider the following 4×7 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

   c. Illustrate the rank-nullity theorem using the matrix A.

2. Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 16 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

   c. Does the union of the two bases found in parts a) and b) span $\mathbb{R}^3$? Substantiate your answer.

3. Consider the following five vectors:

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_4 = \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

   a. Argue why or why not $S = \{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4, \vec{\boldsymbol{v}}_5\}$ is a linearly independent set of vectors.

   b. If S is not a linearly independent set of vectors, remove as many vectors as necessary to find a basis of its linear span and write the remaining vectors in S as a linear combination of the basis vectors.

4. Find the rank and nullity of the following linear transformations:

   a. The orthogonal projection onto the plane $2x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$.
   b. The reflection in $\mathbb{R}^3$ with respect to the line passing through $(31, 67, 97)$.

5. Compute the matrix of the linear transformation

$$T(\vec{\boldsymbol{x}}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix} \vec{\boldsymbol{x}}, \qquad \text{where } \vec{\boldsymbol{x}} \in \mathbb{R}^3,$$

   with respect to the basis

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

6. Consider the following polynomials:

$$1 + x^2, \qquad x - 2x^3, \qquad (1+x)^2, \qquad x^3 + x.$$

   Argue why or why not this set is a basis of the vector space of polynomials of degree ≤ 3.

7. Find a basis of the vector space of all 2×2 matrices S for which

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} S = S \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$