Cornelis VAN DER MEE, Spring 2008, Math 3330, Exam 2

1. Consider the following 5×7 matrix:

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

   c. Illustrate the rank-nullity theorem using the matrix A.

Answer: Seeking the row reduced echelon form of A, we first reorder the rows of A (switching the first two rows and the third and fourth rows), then subtract the third row from the fourth, and divide the fourth row by 2, i.e.,

$$A \Longrightarrow \begin{pmatrix} 1 & 3 & 6 & 0 & 4 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 3 & 6 & 0 & 4 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \mathbf{1} & 3 & 6 & 0 & 4 & 2 & 0 \\ 0 & \mathbf{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the leading ones are written in boldface. Since they occur in the first, second, fourth, and sixth column, the first, second, fourth, and sixth columns of A form a basis of $\operatorname{Im} A$:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

and hence the rank of A equals 4. We now continue to reduce A to echelon form. We subtract three times the second row from the first, yielding

$$\begin{pmatrix} \mathbf{1} & 3 & 6 & 0 & 4 & 2 & 0 \\ 0 & \mathbf{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 4 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, we add the fourth row to the first and subtract the fourth row from the second, yielding the row reduced echelon form

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 4 & 0 & -2 \\ 0 & \mathbf{1} & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, when solving the linear system with coefficient matrix A and zero right-hand sides with the objective of determining $\operatorname{Ker} A$, the echelon form (with an additional eighth zero column in the augmented matrix) implies that x_3, x_5, and x_7 are parameters in the solution (thus implying that the nullity of A equals 3) and that

$$x_1 = -4x_5 + 2x_7, \quad x_2 = -2x_3 - 2x_7, \quad x_4 = -7x_7, \quad x_6 = 2x_7.$$

The vectors $\vec{\boldsymbol{x}}$ in $\operatorname{Ker} A$ can also be written as

$$\vec{\boldsymbol{x}} = x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} 2 \\ -2 \\ 0 \\ -7 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

where the three column vectors form a basis of $\operatorname{Ker} A$. Since rank plus nullity $(4 + 3 = 7)$ equals the number of columns of A, we are in agreement with the Rank-Nullity Theorem.

2. Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix},$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

   c. Illustrate the rank-nullity theorem using the matrix A.

Answer: Let us reduce A to echelon form by switching rows, subtracting twice the second row from the third, and dividing the third row by -6:

$$A \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 3 & 1 \\ 0 & 0 & \mathbf{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the leading ones are written in boldface. It is now clear that A has rank 3 and that the first columns of A form a basis of $\operatorname{Im} A$. A somewhat simpler basis of $\operatorname{Im} A$ is composed of the following three column vectors:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

To determine $\operatorname{Ker} A$, we continue towards the row reduced echelon form of A by subtracting three times the third row from the second, yielding

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 3 & 1 \\ 0 & 0 & \mathbf{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A).$$

The corresponding linear system with coefficient matrix A and zero right-hand sides has as its solution

$$x_1 = 2x_4, \qquad x_2 = 0, \qquad x_3 = -\frac{1}{3}x_4.$$

Thus the vectors $\vec{\boldsymbol{x}}$ in $\operatorname{Ker} A$ have the form

$$\vec{\boldsymbol{x}} = x_4 \begin{pmatrix} 2 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix},$$

where x_4 is a parameter. The column vector in the right-hand side on its own is a basis of $\operatorname{Ker} A$. Since rank plus nullity $(3+1)$ add up to the number of columns of A, we are in a agreement with the Rank-Nullity Theorem.

3. Consider the following four vectors:

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \ \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -4 \end{pmatrix}, \ \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \ \vec{\boldsymbol{v}}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

a. Argue why or why not $S = \{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4\}$ is a linearly independent set of vectors.

b. If S is not a linearly independent set of vectors, remove as many vectors as necessary to find a basis of its linear span and write the remaining vectors in S as a linear combination of the basis vectors.

Answer: Let us work towards the row reduced echelon form of the 4×4 matrix composed of the four column vectors by switching the first three rows, dividing the third row by 2, and dividing the last row by -4:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & -4 & 4 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Now subtract the first row from the second and the third rows, switch the last two rows, subtract twice the third row from the fourth, interchange the second and third rows, and subtract three times the second row from the third:

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -2 & 3 & 0 \\ 0 & \mathbf{1} & -1 & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the leading ones are written in boldface. Thus the rank of A equals 3 and $\vec{\boldsymbol{v}}_4$ is to be written as a linear combination of $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$. To compute the coefficients in this linear combination, we solve the linear system with coefficient matrix A and zero right-hand sides. Continuing towards the row reduced echelon form of A, we subtract three times the third row from the first and add the third row to the second row, which we follow by adding twice the second row to the first:

$$\begin{pmatrix} \mathbf{1} & -2 & 3 & 0 \\ 0 & \mathbf{1} & -1 & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -2 & 0 & -3 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & 0 & 0 & -1 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the vectors $\vec{\boldsymbol{x}}$ in $\operatorname{Ker} A$ all have the form

$$\vec{\boldsymbol{x}} = x_4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

where x_4 is a parameters. We thus get the linear combination

$$\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_3 + \vec{\boldsymbol{v}}_4 = 0.$$

4. Find the rank and nullity of the orthogonal projection onto the hyperplane $x_1 - x_2 + x_3 - x_4 = 0$ in $\mathbb{R}^4$. Argue why your result is correct. Answer: The hyperplane in $\mathbb{R}^4$ has dimension 3 and hence the projection has rank 3. Its kernel is the line though the origin that is perpendicular to the hyperplane and hence its nullity is 1. These results are in agreement with the Rank-Nullity Theorem, since $3 + 1 = 4$, the order of the square matrix representing the projection. If someone really wants to

compute the projection (unnecessarily), its complementary projection onto the line passing through origin and $(1, -1, 1, -1)$ is

$$I - P = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \Longrightarrow P = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

5. Compute the matrix of the linear transformation

$$T(\vec{\boldsymbol{x}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \vec{\boldsymbol{x}}, \qquad \text{where } \vec{\boldsymbol{x}} \in \mathbb{R}^3,$$

with respect to the basis

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Answer: We should determine the matrix B such that

$$B \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

where

$$T(c_1\vec{\boldsymbol{v}}_1 + c_2\vec{\boldsymbol{v}}_2 + c_3\vec{\boldsymbol{v}}_3) = d_1\vec{\boldsymbol{v}}_1 + d_2\vec{\boldsymbol{v}}_2 + d_3\vec{\boldsymbol{v}}_3.$$

Substitution into the latter identity yields

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} B \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

valid for all constnts c_1, c_2, c_3. Multiplying the two square matrices in the left-hand side and deleting the arbitrary column vectors of constants c_1, c_2, c_3, we get the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} B.$$

Solving this linear system by row manipulation of an augmented matrix, we get

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 & 4 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 4 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}\right),$$

where first we have subtracted the first row from the third and then subtracted twice the second row from the third. As a result,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

Alternatively, B can also be computed as follows:

$$\begin{aligned} B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}. \end{aligned}$$

6. Argue why or why not the set of polynomials

$$1 + x^2, \qquad x - x^3, \qquad 1 - x^2, \qquad x + x^3, \qquad x^4,$$

is a basis of the vector space of polynomials of degree ≤ 4. Answer: With respect to the basis $\{1, x, x^2, x^3, x^4\}$, these five polynomials are represented by the column vectors (of coefficients)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Writing these five columns as a 5×5 matrix, we subtract the first row from the third and add the second row to the fourth, yielding

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is an upper triangular matrix with nonzero diagonal elements and hence an invertible matrix. Consequently, $\{1+x^2, x-x^3, 1-x^2, x+x^3, x^4\}$ is a basis of the vector space of polynomials of degree ≤ 4.

7. Find a basis of the vector space of all 2×2 matrices S for which

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Answer: Writing $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

Hence, $a = d$ and $b = c$. A basis of this vector space is obtained by taking the two matrices S for which $a = d = 1$ and $b = c = 0$, and $a = d = 0$ and $b = c = 1$. Thus

$$S = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Though not required, all bases of this vector space are given by the pair of matrices

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix}, \qquad \begin{pmatrix} a_2 & b_2 \\ b_2 & a_2 \end{pmatrix},$$

where $a_1 b_2 - a_2 b_1 \neq 0$.