Cornelis VAN DER MEE, Spring 2008, Math 3330, Sample Exam 2

Name:................................**Grade**:..........**Rank**:...........
To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Consider the following 4×7 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

   c. Illustrate the rank-nullity theorem using the matrix A.

   Answer: As a basis of $\operatorname{Im} A$, take the first, fourth and fifth columns, so that A has rank 3. The kernel of A is composed of the vectors

$$\vec{\boldsymbol{x}} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -4 \\ 0 \\ 0 \\ -6 \\ -8 \\ 1 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -5 \\ 0 \\ 0 \\ -7 \\ -9 \\ 0 \\ 1 \end{pmatrix},$$

   so that A has nullity 4. Since $3+4=7$ is the number of columns of A, we are in agreement with the Rank-Nullity Theorem.

2. Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 16 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

c. Does the union of the two bases found in parts a) and b) span $\mathbb{R}^3$? Substantiate your answer.

Answer: The second and third columns of A form a basis of $\operatorname{Im} A$, while $\operatorname{Ker} A$ consists of all multiples of the column vector $(1, 0, 0)$. However,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 16 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is not a basis of $\mathbb{R}^3$, since two of these vectors are proportional.

3. Consider the following five vectors:

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_4 = \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix}, \quad \vec{\boldsymbol{v}}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

a. Argue why or why not $S = \{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4, \vec{\boldsymbol{v}}_5\}$ is a linearly independent set of vectors.

b. If S is not a linearly independent set of vectors, remove as many vectors as necessary to find a basis of its linear span and write the remaining vectors in S as a linear combination of the basis vectors.

Answer: One cannot have a basis of $\mathbb{R}^4$ consisting of five vectors. Since the 5×4 matrix composed of the five vectors has rank 4, we can in fact delete any of the five vectors to get a basis of $\mathbb{R}^4$. This can be substantiated by showing that the 4×4 matrix composed of the remaining four vectors is invertible.

4. Find the rank and nullity of the following linear transformations:

a. The orthogonal projection onto the plane $2x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$.

b. The reflection in $\mathbb{R}^3$ with respect to the line passing through $(31, 67, 97)$.

Answer: The problem can be done without doing any calculations. The projection P maps $\mathbb{R}^3$ onto a plane (which has dimension 2) along the line of vectors passing through the origin and perpendicular to the plane. Thus the rank of P is 2 and its nullity is 1, in accordance with the Rank-Nullity Theorem ($2+1 = 3$). The reflection R satisfies $R^2 = I$ and hence R is represented by an invertible 3×3 matrix (with inverse R itself). Thus its rank is 3 and its nullity is 0, in agreement with the Rank-Nullity Theorem ($3 + 0 = 3$).

5. Compute the matrix of the linear transformation

$$T(\vec{\boldsymbol{x}}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix} \vec{\boldsymbol{x}}, \qquad \text{where } \vec{\boldsymbol{x}} \in \mathbb{R}^3,$$

with respect to the basis

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Answer: The problem is to find the matrix B satisfying

$$B \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

if

$$T(c_1\vec{\boldsymbol{v}}_1 + c_2\vec{\boldsymbol{v}}_2 + c_3\vec{\boldsymbol{v}}_3) = d_1\vec{\boldsymbol{v}}_1 + d_2\vec{\boldsymbol{v}}_2 + d_3\vec{\boldsymbol{v}}_3.$$

The latter can be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix} B \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

for any $c_1, c_2, c_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} B &= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -7 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 11 & 4 & 0 \\ 1 & -13 & 6 \end{pmatrix}. \end{aligned}$$

6. Consider the following polynomials:

$$1 + x^2, \qquad x - 2x^3, \qquad (1 + x)^2, \qquad x^3 + x.$$

Argue why or why not this set is a basis of the vector space of polynomials of degree ≤ 3. Answer: Let $\{1, x, x^2, x^3\}$ be the usual basis of the vector space V of polynomials of degree ≤ 3. Then $\{1 + x^2, x - 2x^3, (1 + x)^2, x^3 + x\}$ can be written with respect to this "usual" basis as the column vectors

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{\boldsymbol{v}}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We get a basis of V if and only if $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4\}$ is a basis of $\mathbb{R}^4$. The latter is true if and only the 4×4 matrix with these four columns is invertible (i.e., has hank 4), which requires us to show that its echelon form is the 4×4 identity matrix. However, its row reduced echelon form equals

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and hence its kernel consists of all multiples of the column vector $(3, 2, -3, 4)$. Consequently,

$$3(1 + x^2) + 2(x - 2x^3) - 3(1 + x)^2 + 4(x^3 + x) = 0,$$

and therefor $\{1 + x^2, x - 2x^3, (1 + x)^2, x^3 + x\}$ is not a basis of V.

7. Find a basis of the vector space of all 2×2 matrices S for which

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} S = S \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Answer: Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the above equation reduces to

$$\begin{pmatrix} a - c & b - d \\ c - a & d - b \end{pmatrix} = \begin{pmatrix} a + b & a + b \\ c + d & c + d \end{pmatrix},$$

or

$$a - c = a + b, \quad b - d = a + b, \quad c - a = c + d, \quad d - b = c + d.$$

This amounts to $b = -c$ and $a = -d$. Thus S has the form

$$S = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus a basis of this vector space is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$