3380-3 MAKE UP FINAL EXAM Sketched solutions

(1)

$$\begin{matrix} 1 & -7 & 2 & 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & -7 & 2 & 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & -7 & 2 & 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

(rank=5, nullity=2)

then, first

$$\begin{matrix} 1 & -7 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \rightarrow \left.\begin{matrix} 1 & -7 & 0 & 0 & -7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right\} \text{echelon form}$$

(2)

$$\begin{matrix} 1 & 3 & 9/2 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{matrix} \rightarrow \begin{matrix} 1 & 3 & 0 & 23/2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 & -19/2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{matrix}$$

$$x_1 = \frac{19}{2} x_4, \; x_2 = -7 x_4, \; x_3 = 3 x_4 .$$

<u>check</u>: $$\begin{pmatrix} 2 & 6 & 9 & -4 \\ 0 & 0 & 3 & -9 \\ 0 & 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} 19/2 \\ -7 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \underline{\text{OK}}$$

(3)
$$\begin{matrix} 1 & 0 & 0 & -7 \\ 0 & 1 & \cancel{0} & \cancel{0} \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 3 & -8 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{aligned} x_1 &= 7x_4 \\ x_2 &= 0 \\ x_3 &= \frac{8}{3}x_4 \end{aligned}$$

$$\operatorname{Ker} A = \operatorname{span}\left[\begin{pmatrix} 7 \\ 0 \\ 8/3 \\ 1 \end{pmatrix}\right] \qquad \operatorname{Im} A = \operatorname{span}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 1^{st}\text{ column} & 2^{nd}\text{ column} & \frac{3^{rd}\text{ column}}{3} \end{array}$$

(4) With respect to $\{x^4, x^3, x^2, x, 1\}$,

$$\begin{pmatrix} 5 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Lower triangular with nonzero diagonal entries $\Rightarrow$ invertible. Basis

(6)
$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 25 \end{pmatrix} \rightarrow \vec{w}_1 = \vec{v}_1 \rightarrow \vec{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{w}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} - 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} \longrightarrow u_2 = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix}$$

$$\vec{w}_3 = \begin{pmatrix} 0 \\ -1 \\ -4 \\ 0 \end{pmatrix} - 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \left(-\frac{16}{5}\right)\begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 48/25 \\ -1 \\ -36/25 \\ 0 \end{pmatrix}$$

$$48^2 + 25^2 + 36^2 = 2304 + 625 + 1296 = 4225 = 65^2$$

$$\vec{u}_3 = \frac{5}{13}\begin{pmatrix} 48/25 \\ -1 \\ -36/25 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix}1&1&2\\2&3&1\end{pmatrix}\begin{pmatrix}1&2\\1&3\\2&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&1&2\\2&3&1\end{pmatrix}\begin{pmatrix}1\\0\\-3\end{pmatrix}$$

$$\begin{pmatrix}6&7\\7&14\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-5\\-1\end{pmatrix}\rightarrow\begin{pmatrix}x_1\\x_2\end{pmatrix}=\frac{1}{35}\begin{pmatrix}14&-7\\-7&6\end{pmatrix}\begin{pmatrix}-5\\-1\end{pmatrix}=\begin{pmatrix}-63/35\\29/35\end{pmatrix}=\begin{pmatrix}-9/5\\29/35\end{pmatrix}$$

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$$\vec{v}_1=\begin{pmatrix}16\\-12\\21\end{pmatrix}\Rightarrow\vec{w}_1\rightarrow\|\vec{w}_1\|=\sqrt{256+144+441}=\sqrt{841}=29$$

$$\vec{u}_1=\begin{pmatrix}16/29\\-12/29\\21/29\end{pmatrix}$$

$$\vec{w}_2=\begin{pmatrix}4\\3\\0\end{pmatrix}-\left(\frac{64}{29}-\frac{36}{29}\right)\begin{pmatrix}16/29\\-12/29\\21/29\end{pmatrix}=\begin{pmatrix}2916/29^2\\2859/29^2\\-588/29^2\end{pmatrix}\rightarrow\|\vec{w}_2\|=\sqrt{\frac{17022681}{29^2}}=\frac{3}{29}\sqrt{1891409}$$

$$Q=\begin{pmatrix}16/29&2916/\sqrt{17022681}\\-12/29&2859/\sqrt{17022681}\\21/29&-588/\sqrt{17022681}\end{pmatrix}\quad R=\begin{pmatrix}29&28/29\\0&\frac{1}{29}\sqrt{17022681}\end{pmatrix}$$

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$$\det A=2\begin{vmatrix}2&1&-3\\5&4&-6\\8&7&-9\end{vmatrix}=-6\begin{vmatrix}2&1&1\\5&4&2\\8&7&3\end{vmatrix}=-6(24+16+35-32-15-28)=0$$

↳ A^{-1} does not exist.

(10) $\det(\lambda I - A) = \begin{vmatrix} \lambda-6 & 1 \\ 4 & \lambda-9 \end{vmatrix} = \lambda^2 - 15\lambda + 50 = (\lambda-5)(\lambda-10)$

$\underline{\lambda=5}$ $\quad \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathrm{Ker}(5I - A) = \mathrm{span}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

$\underline{\lambda=10}$ $\quad \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathrm{Ker}(10I - A) = \mathrm{span}\left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right)$

$$\underbrace{\begin{pmatrix} 6 & -1 \\ -4 & 9 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}}_{S}\underbrace{\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}}_{D}$$

(11) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 9 & 0 & 10 & \lambda \end{vmatrix} = \lambda \underbrace{\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 10 & \lambda \end{vmatrix}}_{\lambda(\lambda^2+10)} - 9 \underbrace{\begin{vmatrix} -1 & 0 & 0 \\ \lambda & -1 & 0 \\ 0 & \lambda & -1 \end{vmatrix}}_{-1}$

$= \lambda^4 + 10\lambda^2 + 9 = (\lambda^2+1)(\lambda^2+9) \rightarrow \lambda = \pm i, \pm 3i$

Diagonalizable, because A has 4 distinct eigenvalues

(12) $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; A\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 2x-1 \\ 3 \end{pmatrix} = 3\begin{pmatrix} x \\ 1 \end{pmatrix} \rightarrow x = -1 \rightarrow A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$x(n) = A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + A^n\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3^n\begin{pmatrix} -1 \\ 1 \end{pmatrix}$