Cornelis VAN DER MEE, Spring 2008, Math 3330, Final Exam

Name:. . . . .**Grade**:. . . . .**Rank**:. . . . .

1. Bring the following matrix to reduced row echelon form:

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 & 2 & 3 \\ 1 & -3 & 2 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 0 & 6 \end{pmatrix},$$

and determine its rank and nullity. Solution: Switch the first two rows and then the second and the third rows, and then the last two rows:

$$\begin{pmatrix} \mathbf{1} & -3 & 2 & 0 & 1 & 7 \\ 0 & 0 & \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & -1 & 2 & 3 \end{pmatrix},$$

where we have written the leading one's in boldface. Then subtract the second row from the third row, then add the third row to the fourth row, and then divide the last row by 6:

$$\begin{pmatrix} \mathbf{1} & -3 & 2 & 0 & 1 & 7 \\ 0 & 0 & \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -1 & 2 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -3 & 2 & 0 & 1 & 7 \\ 0 & 0 & \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

Now subtract twice the second row from the first row, then subtract the third row from the second row to get

$$\begin{pmatrix} \mathbf{1} & -3 & 0 & -2 & -3 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -3 & 0 & -2 & -3 & 1 \\ 0 & 0 & \mathbf{1} & 0 & 4 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

Then add twice the third row to the first row, and then subtract seven times the last row from the first row and three times the last row from the third row:

$$\begin{pmatrix} \mathbf{1} & -3 & 0 & 0 & -7 & 7 \\ 0 & 0 & \mathbf{1} & 0 & 4 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -3 & 0 & 0 & -7 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 4 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

yielding the row reduced echelon form. Hence the rank of A equals 4 and its nullity equals 2.

2. Find **all** solutions of the linear system $A\vec{\boldsymbol{x}} = \mathbf{0}$, where

$$A = \begin{pmatrix} 3 & 6 & 7 & -3 \\ 0 & 0 & 2 & -8 \end{pmatrix}.$$

Solution: Divide the first row by 3 and the second row by 2 to get

$$\begin{pmatrix} 1 & 2 & \frac{7}{3} & -1 \\ 0 & 0 & 1 & -4 \end{pmatrix}.$$

Then subtract $\frac{7}{3}$ times the second row from the first row to arrive at the row reduced echelon form

$$\begin{pmatrix} \mathbf{1} & 2 & 0 & \frac{25}{3} \\ 0 & 0 & \mathbf{1} & -4 \end{pmatrix},$$

where the leading one's are written in boldface. The solution is as follows:

$$x_1 = -2x_2 - \frac{25}{3}x_4, \qquad x_3 = 4x_4.$$

3. Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 1 & 2 & -8 \\ 1 & 0 & 0 & -5 \\ 0 & 0 & 2 & -8 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.

   b. Find a basis of the kernel of A and show that it really is a basis.

Solution: 1) Switch the first two rows and then the last two rows, and then switch the second and third rows, 2) then the second row by 3, and 3) then subtract both the second and the fourth row from the third row:

$$\begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & -8 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -8 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 2 & -8 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 \end{pmatrix}.$$

Next, switch the last two rows, and then divide the last row by 2:

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & -5 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where in the resulting row reduced echelon form the leading one's have been written in boldface. Since the leading one's occur in the first three rows, we have $\operatorname{rank}(A) = 3$ and

$$\operatorname{Im}(A) = \operatorname{span} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right],$$

where we have divided the third column of A by 2. From the above echelon form we get $\vec{\boldsymbol{x}} \in \operatorname{Ker} A$ if and only if

$$x_1 = 5x_4, \qquad x_2 = 0, \qquad x_3 = 4x_4.$$

Thus the nullity of A equals 1 and

$$\operatorname{Ker}(A) = \operatorname{span} \left[\begin{pmatrix} 5 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right].$$

4. Argue why or why not the set of polynomials

$$35x^4 - 30x^2 + 3, \qquad 5x^3 - 3x, \qquad 3x^2 - 1, \qquad x, \qquad 1,$$

is a basis of the vector space of polynomials of degree ≤ 4. Solution: Writing the five polynomials with respect to the basis $\{1, x, x^2, x^3, x^4\}$ of the vector space V of (real) polynomials of degree ≤ 4, we get the five column vectors

$$\begin{pmatrix} 3 \\ 0 \\ -30 \\ 0 \\ 35 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These five column vectors make up the 5×5 matrix

$$\begin{pmatrix} 3 & 0 & -1 & 0 & 1 \\ 0 & -3 & 0 & 1 & 0 \\ -30 & 0 & 3 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 35 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Switching the first and fifth rows and then the second and third rows we get

$$\begin{pmatrix} 35 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ -30 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 1 \end{pmatrix},$$

which is a lower triangular matrix with nonzero diagonal elements and hence an invertible matrix. Consequently, the original five polynomials form a basis of V.

6. Apply the Gram-Schmidt process to the given basis vectors of

$$V = \text{span}\left[\begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}\right]$$

to obtain an orthonormal basis of V. Solution: Denote the three basis vectors by $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$. We get $\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{v}}_1$ and $\|\vec{\boldsymbol{w}}_1\| = 5$. Thus

$$\vec{\boldsymbol{u}}_1 = \frac{\vec{\boldsymbol{w}}_1}{\|\vec{\boldsymbol{w}}_1\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Next,

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - \underbrace{(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)}_{=0} \vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_2,$$

so that $\|\vec{\boldsymbol{w}}_2\| = \sqrt{0^2 + 4^2 + 3^2 + 0^2} = 5$. Hence

$$\vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{w}}_2}{\|\vec{\boldsymbol{w}}_2\|} = \begin{pmatrix} 0 \\ 4/5 \\ 3/5 \\ 0 \end{pmatrix}.$$

$$\vec{\boldsymbol{w}}_3 = \vec{\boldsymbol{v}}_3 - \underbrace{(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1)}_{=0} \vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{v}}_3 - \underbrace{(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2)}_{=16/5} \vec{\boldsymbol{u}}_2 = \vec{\boldsymbol{v}}_3 - \frac{16}{5}\vec{\boldsymbol{u}}_2 = \begin{pmatrix} 0 \\ 36/25 \\ -48/25 \\ 1 \end{pmatrix}.$$

Next,

$$\|\vec{\boldsymbol{w}}_3\| = \sqrt{0^2 + (36/25)^2 + (-48/25)^2 + 1^2} = \frac{13}{5}.$$

Hence

$$\vec{\boldsymbol{u}}_3 = \frac{\vec{\boldsymbol{w}}_3}{\|\vec{\boldsymbol{w}}_3\|} = \begin{pmatrix} 0 \\ 36/65 \\ -48/65 \\ 5/13 \end{pmatrix}.$$

7. Find a least-squares solution to the system

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

Solution: Writing the above system as $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, we now write down

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}_{=A^TA} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{=\vec{\boldsymbol{x}}} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}}_{=A^T\vec{\boldsymbol{b}}}.$$

OR:

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Consequently, the least squares solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

8. Find the factors Q and R in the QR factorization of the matrix

$$M = \begin{pmatrix} 12 & 0 \\ 3 & 4 \\ 4 & -3 \end{pmatrix}$$

by using the Gram-Schmidt process. Solution: Denote the columns of M by $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. Then $\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{v}}_1$ and $\|\vec{\boldsymbol{w}}_1\| = \sqrt{12^2+3^2+4^2} = 13$. Thus the first orthonormal basis vector is given by

$$\vec{\boldsymbol{u}}_1 = \begin{pmatrix} 12/13 \\ 3/13 \\ 4/13 \end{pmatrix}.$$

We now compute

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_2,$$

because $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are orthogonal. Since $\|\vec{\boldsymbol{w}}_2\| = \sqrt{0^2+4^2+(-3)^2} = 5$, we get for the second orthonormal basis vector

$$\vec{\boldsymbol{u}}_2 = \begin{pmatrix} 0 \\ 4/5 \\ -3/5 \end{pmatrix}.$$

Hence the QR-factorization of M is given by

$$\underbrace{\begin{pmatrix} 12 & 0 \\ 3 & 4 \\ 4 & -3 \end{pmatrix}}_{=M} = \underbrace{\begin{pmatrix} 12/13 & 0 \\ 3/13 & 4/5 \\ 4/13 & -3/5 \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 13 & 0 \\ 0 & 5 \end{pmatrix}}_{=R}.$$

9. Find the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Explain why or why not A^{-1} exists. Solution:

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix},$$

where we have subtracted four times the first row from the second row and seven times the first row from the third row. We have now obtained a matrix, where the third row is twice the second row. Thus $\det(A) = 0$ and hence A^{-1} does not exist.

10. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & -3 \\ -2 & 7 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: We have

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 7 \end{vmatrix} \\ &= (\lambda - 6)(\lambda - 7) - 6 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9). \end{aligned}$$

Thus the eigenvalues are $\lambda = 4$ and $\lambda = 9$. We now compute

$$\begin{aligned} \lambda = 4: \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(4I - A) = \mathrm{span}\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right], \\ \lambda = 9: \quad \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(9I - A) = \mathrm{span}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]. \end{aligned}$$

Hence the diagonalization of A is as follows:

$$\underbrace{\begin{pmatrix} 6 & -3 \\ -2 & 7 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}}_{=D}.$$

11. Find all eigenvalues (real and complex) of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 4 & -2 & 0 \end{pmatrix}.$$

Why or why not is it possible to diagonalize the matrix A? Solution: We have

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ -3 & 0 & \lambda & -2 \\ 0 & -4 & 2 & \lambda \end{vmatrix} \\ &= \lambda \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -2 \\ -4 & 2 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ -3 & \lambda & -2 \\ 0 & 2 & \lambda \end{vmatrix} \\ &= \lambda^2(\lambda^2 + 4) + 1(\lambda^2 + 4) = (\lambda^2 + 1)(\lambda^2 + 4). \end{aligned}$$

Thus the eigenvalues of A are $\pm i$ and $\pm 2i$. Since these four eigenvalues are distinct, the matrix A is diagonalizable.

12. Consider the discrete dynamical system

$$x(n+1) = Ax(n), \qquad n = 0, 1, 2, 3, \dots,$$

where

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

a. Write $x(1) = Ax(0)$ as a multiple of an eigenvector of A.

b. Compute the solution $x(n)$ for $n = 1, 2, 3, \dots$.

Solution: The matrix A has trace $2 + (-2) = 0$ and determinant 0. So there is an eigenvalue 0 with algebraic multiplicity 2. We also have

$$\operatorname{Ker} A = \operatorname{span}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right],$$

which makes A nondiagonalizable. Thus

$$Ax(0) = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and hence

$$A^2x(0) = 2A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$x(n) = A^n x(0) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & n = 0, \\ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, & n = 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & n = 2, 3, 4, \dots. \end{cases}$$